An introduction and analysis of the De Donder-Weyl Hamiltonian Formalism in the context of Spin-2 Theories

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Abstract

As research dives deeper into the foundations of our physical world, physical theories become more sophisticated and mathematically complex. New methods have to be developed to analyse and understand the results uncovered. A prevalent tool to inspect covariant field theories is the Hamiltonian formalism, which requires the singling out of a specific time direction to compute the canonical momenta. The polysymplectic De Donder-Weyl Hamiltonian formalism [1, 2, 3] may be a covariant alternative to the classical Hamiltonian.

We review General Relativity and Bimetric Theory [4] and introduce the De Donder-Weyl Hamiltonian, with a focus on practical application to existing theories. We compute the equations of motion for a $SO(1,5)$ gauge theory inspired by MacDowell and Mansouri's construction [5]. It is shown that, in contrast to the traditional formalism, the point during the calculation at which the $SO(1,5)$ symmetry is broken has no influence on the result. We also introduce the constraint analysis in terms of the new formalism and apply it to the action of Bimetric Theory up to the computation of the complete set of secondary constraints.

Contents

Introduction

The two most important theories describing the fundamental physics of our world are the Standard Model of particle physics (SM) and General Relativity (GR). While the SM applies on the smallest scales like the electroweak theory and strong interactions, GR works on the biggest scales and spawned theories like cosmology, which describes the largest-scale structures in the universe and their dynamics.

In the last 100 years physicists have been able to observe and thoroughly test the predictions made by GR, putting the theory on a solid foundation. This requires any modified theory to make the same predictions as GR to be considered physical. Describing some observations in a consistent theoretical framework requires going beyond GR. One observation of note is the acceleration of the universe. This behaviour can be captured by adding a cosmological constant to the equation of motion (EoM) of GR. The origin of this constant, however, cannot be derived in unmodified GR. Another interesting observation that might require an extension of GR is dark matter. There are galaxies in the universe which, according to GR, would not be able to form or hold together and therefore require additional matter, dark matter, to interact with the visible matter to be stable. These anomalies and other factors led physicists to consider extended theories of gravity.

Like the SM, GR can be formulated as a relativistic field theory. Where in the SM the fields are of spin-0, 1/2 and 1, the field of GR is a massless spin-2 field. A natural extension of such a massless spin-2 theory is the addition of a massive spin-2 field. To a linear level, a massive spin-2 theory has first been inspected by Fierz and Pauli [6]. A covariant action of a fixed massless and a dynamic massive spin-2 field was first believed to be impossible due to the appearance of a ghost [7]. However, in [8, 9] it was shown that by promoting both metrics to the dynamic status the ghost is eliminated and the action is covariant. This theory is called Bimetric Theory (BT) [4]. In [10] the authors formulated BT in a gauge theoretic way based on the $SO(1,5)$ group, inspired by the construction of MacDowell and Mansouri [5]. A similar action to BT was obtained by taking advantage of a local splitting of $SO(1,5)$, breaking the symmetry down to $SO(1,3) \times SO(2)$.

A classic approach to evaluate such theories and their physicality, i.e. whether they contain the correct amount of degrees of freedom (DoF) or propagate a ghost, is the Hamiltonian constraint analysis introduced by Dirac [11]. In this analysis one starts out with the Hamiltonian of the theory and computes its constraints in an iterative way. This approach has been used by a lot of physicists to inspect the validity of their theories and has been a useful tool in their toolbox. But as theories grow more complex, their tools have to grow as well. GR is formulated in a covariant way, i.e. the action does not provide a preferred time direction. To do a Hamiltonian analysis, however, one must choose such a direction to be able to compute the canonical momenta. The De Donder-Weyl (DDW) Hamiltonian formalism [1, 2, 3] is an extension to the traditional Hamiltonian formalism, where this covariance is retained. Although this formalism has been known since the 1930's it only recently started gaining popularity [12]. In this formulation, the analogue to the canonical momenta are the polymomenta, which are partial derivatives of the Lagrangian density in all space-time directions instead of just the time direction. This gives the DDW Hamiltonian some interesting properties, which are not present in the traditional Hamiltonian.

In this work we will apply the DDW formalism to the $SO(1,5)$ gauge theory of [10] and inspect the constraints of BT. In the former we uncover that, we were able to apply the DDW formalism to both the $SO(1,3)\times SO(2)$ action from [10] and the full $SO(1,5)$ action, projecting the EoM down to $SO(1,3)\times SO(2)$, and still arrive at the same results. The constraint analysis of BT is done in an explicit and detailed manner for the primary constraints and the consistency conditions, following the approach introduced in [13].

Understanding the DDW formalism requires a solid foundation of differential geometry that goes beyond the level taught in graduate physics classes. Therefore this thesis aims to break it down to its most essential parts and still give a comprehensive introduction to the formalism. With every new tool in a tool kit, it is necessary to evaluate its validity and usefulness. Obviously, every formalism is absolutely required to deliver valid physical results, which we assume for a given, regarding the literature ([14, 15, 16] to name a few). One of the requirements for this thesis is, that it should be written in a way so that it can be understood by a graduate student. This gives us a natural lens through which to judge the formalism. We will answer the question, whether this formalism could be taught in graduate classes to enhance a students tool kit. The search for the answer to this question will not get its own section in this thesis. It can rather be seen as a sub-theme or a common thread weaving itself through this work.

The first half of this thesis will introduce the physical and mathematical background, necessary to understand the second half, where we will work with the actual formalism and theories mentioned. In chapter 1, we will introduce several spin-2 theories, starting with GR and its vielbein formulation and ending with BT. Chapter 2 contains the MacDowell-Mansouri approach to a gauge theory of GR and the $SO(1,5)$ theory of [10]. In chapter 3 we will get to know the Hamiltonian formalism and the constraint analysis introduced by Dirac and apply it to a short example in the context of GR. Following that, in chapter 4 we introduce the DDW formalism together with a short application of a system of real scalar fields. Chapter 5 sees the application of the DDW formalism to the $SO(1,5)$ theory previously introduced. The final chapter 6, before the conclusion and outlook in 7, is the constraint analysis of BT.

1 Spin-2 Theories

This chapter will give an overview of General Relativity (GR) and two other popular spin-2 field theories. Those two theories are massive gravity (section 1.3) and bimetric gravity (section 1.4).

1.1 Relativistic Field theories

The two pillars on which our understanding of physics rests today are Einstein's theory of general relativity, which describes all gravitational phenomena, and the standard model of particle physics (SM), which covers everything else. Both theories are formulated as field theories. Such theories describe the world through the dynamics and interactions of *fields*. These fields are classified in the Wigner classification [17] by their spin and mass, where the spin of a field is determined by its representation. For example, a scalar field would have spin 0 or a vector field would be of spin 1. The physical degrees of freedom (DoF) of a field are determined by its mass (or masslessness) and spin. A massive field always has $2s + 1$ DoF, with s being the spin, whereas a massless field has always only 2 DoF (except for the real scalar field, which has 1 DoF).

There exist consistent theories of spin 0, 1/2 and 1 for both massive and massless fields. When quantized, these fields and their interactions become the foundational parts making up the standard model. To formulate such a theory, one usually starts at the field Langrangian and applies the principle of least action (see chapter 3) to arrive at the equations of motion describing the dynamics of such a field. As an example consider the covariant formulation of Maxwell theory, describing the vector potential A_μ. The source-free Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1.1}$$

where $F_{\mu\nu}$ is the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The equation of motion is

$$\partial_\mu F^{\mu\nu} = 0. \tag{1.2}$$

But A_μ has 4 free functions, while the theory should only have 2 DoF. Is there something wrong? No, but we still have some *gauge fixing* left to do. First, by looking at the definition of $F_{\mu\nu}$, we see that it is invariant under the transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \varphi$ for any differentiable scalar function $\varphi(x)$. We can fix one component of A_μ by choosing φ, such that $\partial^\mu \partial_\mu \varphi = -\partial^\mu A_\mu$. By doing this (1.2) becomes

$$\partial_\mu \partial^\mu A^\nu = 0.$$

We are left with 3 DoF and are able to fix another component of A_μ by introducing a field solving $\partial_\mu \partial^\mu \phi = 0$ and repeating a similar process to the one outlined above. This is called the *Lorenz gauge* and is one popular way of fixing the DoF that are left over.

So far, the process of finding constraints and gauge fixing to remove the unphysical DoF of a theory has just been roughly sketched out, but one can already tell that it can become quite complex for fields with a higher number of DoF.

The metric $g_{\mu\nu}$ in GR is the massless field of the classic theory of a spin-2 field and is a case where the number of free parameters is much higher (10) than the actual physical DoF (2). The natural question to ask is whether there exist *massive* spin-2 fields and we will go into more detail on that in sections 1.3 and 1.4.

When talking about DoF and physicality of a theory, one has to mention *ghosts*. Ghosts are fields with a negative kinetic term. A theory must be ghost-free to be physical, since otherwise the Hamiltonian would not be bounded from below. One example of such a field is the *Boulware-Deser ghost* [7, 18], which we will look at in more detail in section 1.3.

1.2 General Relativity

1.2.1 Introduction

The most well-known spin-2 theory is Einstein's theory of GR [19]. It is a geometric theory of gravity, where gravity manifests itself in the curvature of 4-dimensional spacetime. It is formulated as a field theory of the spin-2 field $g_{\mu\nu}$, which is the metric on the manifold M we call spacetime. The curvature on M is described by the *Riemann tensor* given in terms of the *Levi Civita connection* ∇ and can be expressed as

$$R^{\alpha}{}_{\beta\mu\nu}v^{\beta} = [\nabla_{\mu}, \nabla_{\nu}]v^{\alpha},$$

for any vector v^{α}, where $[a, b] = (ab - ba)$ is the *commutator*. Vectors that are parallely transported in a loop on curved space tend to change. This change is described by the Riemann tensor and is, for an area $\delta S^{\gamma\delta}$, given by

$$\delta v^{\alpha} = R^{\alpha}_{\beta\gamma\delta}\delta S^{\gamma\delta}v^{\beta}.$$

The connection, or covariant derivative, is unique and given by $\nabla_{\alpha}g_{\mu\nu} = 0$. For the arbitrary vector v^{α} the covariant derivative is defined as

$$\nabla_{\alpha}v^{\beta} = \partial_{\alpha}v^{\beta} + \Gamma^{\beta}_{\alpha\gamma}v^{\gamma},$$

where the $\Gamma^{\beta}_{\alpha\gamma}$ are called *Christoffel symbols*. They are defined as

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(\partial_{\beta}g_{\gamma\delta} + \partial_{\beta}g_{\alpha\delta} - \partial_{\delta}g_{\beta\gamma}). \tag{1.3}$$

The above definition of the Christoffel symbols is a consequence of spacetime being *torsion-free*. Torsion is described by the *torsion tensor* and can be envisioned as the 'twist' of the tangent vector of a curve as the curve evolves. By contracting two indices of the the Riemann tensor we obtain the *Ricci tensor*

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}.$$

Contracting this tensor with the metric $g^{\mu\nu}$, we get the *Ricci scalar*

$$R = g^{\mu\nu}R_{\mu\nu},$$

which is used in the construction of the *Einstein-Hilbert action*

$$\mathcal{S}_{EH} = m_P^2 \int \mathrm{d}^4x\sqrt{-g}(R - 2\Lambda), \tag{1.4}$$

where m_P is the reduced Planck mass defined as $m_P^2 = \frac{\hbar c}{16\pi G}$ and Λ is the cosmological constant. In the above definition we follow the convention of writing $g = \det g_{\mu\nu}$. In the presence of matter fields the total action becomes

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_M. \tag{1.5}$$

Varying (1.5) with respect to the metric we are able to compute *Einsteins field equations*

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{m_P^2} T_{\mu\nu}, \tag{1.6}$$

where $T_{\mu\nu}$ is the *energy-momentum tensor* we gain by varying $\mathcal{S}_M$ and $G_{\mu\nu}$ is called the *Einstein tensor*. The Einstein tensor satisfies the Bianchi identities

$$\nabla^\mu G_{\mu\nu} = 0,$$

which implies local energy-momentum conservation $\nabla^\mu T_{\mu\nu} = 0$. Comparing (1.6) and the EoM (1.2), we see that the Einstein equations are not linear in the metric $g_{\mu\nu}$. This means that the field will interact with itself and give rise to a non-linear theory, leading to problems in formulations of *massive gravity*, see section 1.3.

1.2.2 The 3+1 formalism

The Hamiltonian formalism, which will be introduced in chapter 3, requires the singling out of a specific coordinate in $\mathcal{L}$ as the time coordinate (for example, we need it to define the canonical momenta). To accomplish that we need to find a useful splitting of the metric $g_{\mu\nu}$. This was done by foliating spacetime into a set of spacelike hypersurfaces $\Sigma \subset M$, which share a common orthogonal time direction [20]. The metric can then be decomposed into

$$g = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ N_i & \gamma_{ij} \end{pmatrix}, \tag{1.7}$$

where N is called the *lapse*, N_i is a 3-vector on Σ called the *shift* and γ_{ij} is the induced metric obtained by restricting g to Σ. The inverse of g in the 3+1 decomposition takes on the form

$$g^{-1} = N^{-2}\begin{pmatrix} -1 & N^j \\ N^i & N^2\gamma^{ij} - N^i N^j \end{pmatrix}.$$

Using the above decomposition (1.7), we can express the action (1.5) in terms of N, N_i and γ_{ij}. The action does not depend on time derivatives of either the lapse or the shift. Therefore, the only dynamical variables are the components of γ_{ij} and their canonical momenta π^{ij} defined as

$$\pi^{ij} = \frac{\partial \mathcal{L}_{EH}}{\partial \dot{\gamma}_{ij}} = \frac{m_P^2\sqrt{\gamma}}{2N}\Big(2\gamma^{i[k}\gamma^{j]l}\dot{\gamma}_{ij} - \nabla^i N^j - \nabla^j N^i + 2\gamma^{ij}\nabla^k N_k\Big),$$

where $\dot{\gamma}$ denotes a time derivative, $\gamma^{i[k}\gamma^{j]l} = \frac{1}{2}(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})$ and ∇^i is the covariant derivative compatible with γ_{ij}. The Einstein-Hilbert action without matter action or cosmological constant can then be expressed as

$$\mathcal{S}_{EH} = \int \mathrm{d}^4x\big(\pi^{ij}\dot{\gamma}_{ij} + N\mathcal{C} + N^i\mathcal{C}_i\big), \tag{1.8}$$

where

$$\mathcal{C} = m_P^2\sqrt{\gamma}\mathcal{R}(\gamma) + \frac{1}{m_P^2\sqrt{\gamma}}\left(\frac{\pi^2}{2} - \pi^{ij}\pi_{ij}\right) \quad \text{and}$$
$$\mathcal{C}_i = 2\sqrt{\gamma}\nabla^j\pi_{ij}$$

with $\mathcal{R}(\gamma)$ being the Ricci scalar associated with γ.

1.2.3 The vielbein formalism

The metric is not the only way to mathematically represent a spin-2 field. In all calculations and definitions above, we have worked in the so called *holonomic basis*, i.e. ∂_μ for the tangent space and $\mathrm{d}x^\mu$ for the cotangent space. But this is not the only choice of basis we can make. Let us choose a set of basis vectors $e_A = e_A^{\ \mu}\partial_\mu$ and one-forms $e^A = e^A_{\ \mu}\mathrm{d}x^\mu$ where $A = 0, 1, 2, 3$, such that

$$g(e_A, e_B) = e^\mu_A g_{\mu\nu} e^\nu_B = \eta_{AB} \quad \text{and the same for the inverse}$$
$$g^{-1}(e^A, e^B) = e^A_\mu g^{\mu\nu} e^B_\nu = \eta^{AB}.$$

Because we know that $\partial_\mu \mathrm{d}x^\nu = \delta^\nu_\mu$, we can show that e^μ_A and e^A_μ are inverses of each other:

$$e_A e^B = e^\mu_A e^B_\nu \partial_\mu \mathrm{d}x^\nu = e^\mu_A e^B_\mu = \delta^B_A.$$

This construction is called *vielbein* or *tetrad* (or *vierbein* in this case [1]) and can be represented as an invertible matrix with 16 components e^μ_A. They form an orthonormal basis of the tangent and cotangent spaces. This approach was first formulated by Einstein in 1928 in the attempt of unifying gravity and electromagnetism[21].
Although the notation is similar, note that the vielbein is not a tensor. The two separate sets of indices denote that it connects two different vector spaces. The greek indices denote spacetime indices, while the capital latin letters are called Lorentz indices.
The matrix representing the vierbein can always be brought into a lower triangular form, using a Lorentz transformation. In the 3+1 decomposition this takes the form

$$e = \begin{pmatrix} N & 0 \\ E^a_j N^j & E^a_i \end{pmatrix} \quad \text{and } e^{-1} = \frac{1}{N}\begin{pmatrix} 1 & 0 \\ -N^i & N E^i_a \end{pmatrix}, \tag{1.9}$$

where E^a_i is the spatial vielbein obtained by doing the above construction for the metric γ_{ij} (Note that we kept the indices a, b reserved as spatial Lorentz indices). In the vierbein formulation the Einstein-Hilbert action (1.4) becomes

$$\mathcal{S}_{EH} = m_P^2 \int \mathrm{d}^4x \det(e) R(e),$$

where $R(e)$ is the Ricci scalar in terms of e. Using the decomposition (1.9), we arrive at a expression similar to (1.8), namely

[1] vielbein is German and means *manyleg*, a vierbein is a *fourleg*

$$\mathcal{S}_{EH} = \int \mathrm{d}^4x \bigg(\pi(E)^i_a \dot{E}^a_i - N\mathcal{C} - N^i\mathcal{C}_i - \frac{1}{2}\lambda^{ab}\mathcal{P}_{ab} \bigg), \tag{1.10}$$

where $\mathcal{P}_{ab}$ are related to the spatial part of the Lorentz transformation. A detailed computation of (1.10) is found in [22] and is not very relevant for understanding the concept. What is relevant though is the fact the we were able to single out specific parts of the metric $g_{\mu\nu}$, pinpoint where the dynamics lie and simplify the representation of the action $\mathcal{S}_{EH}$. Another advantage of this formulation is that some of the calculations, like counting the DoF, are easier, although we pick up an extra constraint in the form of $\mathcal{P}_{ab}$. This is because of the fact that N and N^i take on the shape of Lagrange multipliers for the constraints $\mathcal{C}$ and $\mathcal{C}_i$. One of the disadvantages of this approach is the fact, that it becomes extremely hard, to untangle these equations and go back to the old metric formulation.

1.2.4 The spin connection and the Palatini action

The Christoffel symbols (1.3) are the connection forms of the Levi-Civita connection on the manifold M. With the introduction of the vielbeins there is another connection form called the *spin connection* ω^{AB}_μ, which is necessary to define a covariant derivative acting on the Lorentz indices, namely

$$\mathcal{D}_\mu V_A = \partial_\mu V_A + \omega^B_{\mu A} V_B.$$

The associated curvature 2-form is

$$R^{AB}_{\mu\nu} = 2\partial_{[\mu}\omega^A_{\nu]B} + 2\omega^A_{[\mu C}\omega^{CB}_{\nu]}.$$

With the spin connection there is a more general covariant derivative for tensors like $V = V^A_\mu \mathrm{d}x^\mu \otimes e_A$, i.e.

$$\nabla_\mu V^A_\nu = \partial_\mu V^A_\nu - \Gamma^\alpha_{\mu\nu} V^A_\alpha + \omega^A_{\mu B} V^B_\nu, \tag{1.11}$$

which annihilates the vielbein ($\nabla_\mu e^A_\nu = 0$) and completely determines the spin connection. Treating e and ω as independent fields, the Ricci scalar of GR can now be expressed in terms of e_μ^A and the curvature $R_{\mu\nu}^{AB}$, which yields the Palatini action with a cosmological constant

$$\mathcal{S}_P = \int \mathrm{d}^4x\sqrt{-g}R = \int \mathrm{d}^4x e\big(e_A^\mu e_B^\nu R_{\mu\nu}^{AB} + \Lambda\big), \tag{1.12}$$

where we used the fact that $\sqrt{-g} = e$ [23]. In this formalism, the metric g is not the most fundamental field any more and the relation between the spin connection and the Christoffel symbols is given by

$$\omega_\mu^{AB} = e_\nu^A \Gamma^\nu_{\sigma\mu} e^{\sigma B} - e^{\nu B}\partial_\mu e_\nu^A.$$

One more thing to note here, is that the spin connection is the *gauge field* generated by local Lorentz transformations, which becomes important in chapter 2 where we will inspect several approaches to gravity as a gauge theory.

1.3 Massive Gravity

A theory of massive gravity is constructed by simply adding a mass term for the metric to the action of a massless spin-2 field. We will begin by looking at the approach formulated by Fierz and Pauli [6], which is a linear theory, and afterwards continue on to the non-linear theory.

1.3.1 Fierz-Pauli theory

The linear theory is achieved by expanding the metric around a flat background, i.e. the Minkowski metric. This gives

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{m_P}h_{\mu\nu}, \tag{1.13}$$

where $h_{\mu\nu}$, the perturbation, is the field of interest. We will continue to formulate EoM for h up to linear order. The free action for h up to quadratic order becomes

$$\mathcal{S} = \frac{1}{2}\int \mathrm{d}^4x h_{\mu\nu}\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma},$$

where

$$\mathcal{E}^{\mu\nu\rho\sigma} = -\Big(\eta^{\rho[\nu}\partial^{\sigma]}\partial^{\mu} + \eta^{\mu[\rho}\partial^{\nu]}\partial^{\sigma} + \eta^{\mu[\nu}\eta^{\rho]\sigma}\partial^{2}\Big).$$

The tensor $\mathcal{E}^{\mu\nu\rho\sigma}$ is the *linearized Einstein tensor*. It is obtained by plugging (1.13) and it's inverse $g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}h_{\alpha\beta}\eta^{\beta\nu} + \mathcal{O}(h^2)$ into the expression (1.3) and computing the Einstein tensor from the resulting expression up to linear order in h.
The field equations then become

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} = 0,$$

where we use the Minkowski metric to raise and lower indices.
Similar to (1.1), the action of h is invariant under the infinitesimal spacetime translation

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - 2\partial_{(\mu}\varphi_{\nu)}, \tag{1.14}$$

where we used the commutator in the indices to express $\partial_{(\mu}\varphi_{\nu)} = \frac{1}{2}(\partial_{\mu}\varphi_{\nu} + \partial_{\nu}\varphi_{\mu})$. This symmetry is related to the fact that GR is diffeomorphism invariant and gives us the ability to fix 4 DoF of $h_{\mu\nu}$ using φ_{μ}.
The next step is the addition of a mass term to the action, which yields

$$\mathcal{S}_{FP} = \frac{1}{2}\int \mathrm{d}^4x\Big(h_{\mu\nu}\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma} - \frac{m^2}{2}(h^{\mu\nu}h_{\mu\nu} - ah^2)\Big), \tag{1.15}$$

where m is the mass of the field h, a a dimensionless scaling factor and $h \coloneqq h^{\mu}_{\mu}$. The variation of (1.15) gives us the EoM

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} - \frac{m^2}{2}(h_{\mu\nu} - a\eta_{\mu\nu}h) = 0. \tag{1.16}$$

We see that the symmetry (1.14) is broken by the mass term. To find some constraints we can apply the contracted Bianchi identity $\partial^{\mu}\mathcal{E}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} = 0$, which leads us to the constraints

$$\partial^\mu h_{\mu\nu} - a\partial_\nu h = 0. \tag{1.17}$$

We were able to remove 4 DoF from the 10 propagating modes of the symmetric field $h_{\mu\nu}$. This leaves us with 6 DoF. However, a massive spin-2 field is supposed to have 5 DoF. To find the last constraint, we take the trace of (1.16) and arrive at

$$\partial^\mu(\partial^\nu h_{\mu\nu} - \partial_\mu h) - \frac{m^2}{2}(1-4a)h = 0.$$

The above equation turns into a constraint for the exact case where $a = 1$, because then we can apply (1.17) to eliminate the first term and are left with the fifth constraint $h = 0$. This leaves us with 5 and therefore the correct number of DoF for a massive spin-2 field theory. This is only the case if $a = 1$ otherwise the theory would contain a ghost and would be unphysical.

1.3.2 Nonlinear massive gravity

A mass term for a spin-2 field must be a scalar density, meaning it should be some scalar function $V(g)$ multiplied by the scalar density $\sqrt{-g}$. Therefore, it cannot have any free indices, which is where the first problem lies. We cannot simply construct a similar action to (1.15), as the contraction $g^{\mu\nu}g_{\mu\nu} = 4$ is trivial. De Rahm and Gabadadze concluded from this, that it is not possible to construct a covariant nonlinear action without the introduction of a second tensor field [24, 25] . This could theoretically be any object with enough indices, but the minimal choice is a rank-2 tensor $f_{\mu\nu}$. The introduced field is then used to contract with $g_{\mu\nu}$ and is interpreted as a background metric. A general action is expected to take on the form

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\big[R(g) - m^2V(g,f)\big].$$

The issue with this construction however, as argued in [7] and [18], is the additional sixth DoF referred to as the *Boulware-Deser ghost.* This is due to two reasons. First, by looking at the 3+1 decomposition of g

$$\mathcal{S} = \int \mathrm{d}^4x \big[\pi^{ij}\dot{\gamma}_{ij} + N\mathcal{C} + N^i\mathcal{C}_i - m^2 V(g,f)\big],$$

we see that the lapse and shift are not Lagrange multipliers any longer, if $V(g,f)$ is not linear in either of them. Therefore, their EoM will not function as constraints on γ_{ij} any more. The mass term breaks covariance and we are not able to remove DoF through coordinate transformations. We are not able to constrain any of the 12 DoF propagating in γ_{ij} and π_{ij}.
De Rahm, Gabadze and Tolley were able to construct an action which circumvents the above problem [24, 25]. The resulting theory is called dRGT massive gravity and has the action

$$\mathcal{S}_{dRGT} = \int \mathrm{d}^4x\sqrt{-g}\left[R - 2m^2\sum_{n=0}^{4}\beta_n e_n(S)\right]. \tag{1.18}$$

The matrix S is the square root of $g^{-1}f$ defined as $S^{\mu}_{\nu}S^{\nu}_{\gamma} = g^{\mu\nu}f_{\nu\gamma}$ and the β_n are free parameters of the theory. The $e_n(S)$ are the elementary symmetric polynomials of the eigenvalues of S and defined as

$$e_0(S) = 1, \qquad e_1(S) = [S], \qquad e_2(S) = \frac{1}{2}([S]^2 - [S^2]) \tag{1.19}$$

$$e_3(S) = \frac{1}{6}\big([S]^3 - 3[S][S^2] + 2[S^3]\big) \tag{1.20}$$

$$e_4(S) = \frac{1}{24}\big([S]^4 - 6[S]^2[S^2] + 3[S^2]^2 + 8[S][S^3] - 6[S^4]\big) \tag{1.21}$$

$$= \mathrm{det}S, \tag{1.22}$$

where $[S] = \mathrm{Tr}(S)$. The derivation of the theory was done on a flat background, i.e. for $f_{\mu\nu} = \eta_{\mu\nu}$. Later on, it was shown in [26] that the action is consistent for arbitrary $f_{\mu\nu}$ and β_n and that there exist constraints that remove the Boulware-Deser ghost [27] [8].

1.4 Bimetric Gravity

In their *bimetric theory* Hassan and Rosen [4] promoted the fixed metric $f_{\mu\nu}$ from massive gravity to a dynamic field and showed that the resulting theory is ghost-free.

This would not only make the theory more natural, since f was added into (1.18) and fixed by hand, but also make the theory generally covariant again. With two dynamic metrics the action takes on the form

$$\mathcal{S}_{HR} = \int \mathrm{d}^4x \left[m_g^2 \sqrt{-g} R(g) + m_f^2 \sqrt{-f} R(f) - 2m^4 \sqrt{-g} \sum_{n=0}^{4} \beta_n e_n(S) \right], \tag{1.23}$$

where $m_{f/g}$ are the Planck masses for the respective sector and the rest is similar to (1.18). Looking at the mass term, it seems like the action treats g different from f. But the action is actually symmetric under the exchange of f and g, if we simultaneously exchange $m_f \leftrightarrow m_g$ and $\beta_n \leftrightarrow \beta_{4-n}$, because

$$\sqrt{-g} \sum_{n=0}^{4} \beta_n e_n(\sqrt{g^{-1}f}) = \sqrt{-f} \sum_{n=0}^{4} \beta_n e_{4-n}(\sqrt{f^{-1}g})$$

as shown in [4].
Varying the action (1.23) with respect to f and g yields the EoM

$$\begin{aligned} G_{\mu\nu}(g) &= -\frac{m^4}{m_g^2} V_{\mu\nu}^{(g)} \\ G_{\mu\nu}(f) &= -\frac{m^4}{m_f^2} V_{\mu\nu}^{(f)}, \end{aligned}$$

with $G_{\mu\nu}(f \text{ or } g)$ being the respective Einstein tensor and

$$V_{\mu\nu}^{(g)} = -\frac{2}{\sqrt{-g}} \frac{\partial}{\partial g^{\mu\nu}} \left[\sqrt{-g} \sum_{n=0}^{4} \beta_n e_n(S) \right] \quad \text{and} \tag{1.24}$$

$$V_{\mu\nu}^{(f)} = -\frac{2}{\sqrt{-f}} \frac{\partial}{\partial f^{\mu\nu}} \left[\sqrt{-g} \sum_{n=0}^{4} \beta_n e_{4-n}(S^{-1}) \right]. \tag{1.25}$$

Applying the 3+1 decomposition (see section 1.2.2) to $g_{\mu\nu}$ and $f_{\mu\nu}$, we can decompose the Einstein-Hilbert part of (1.23) into

$$\mathcal{S}_{HR} = \int \mathrm{d}^4x \Big[\pi^{ij}\dot{\gamma}_{ij} + N\mathcal{C}_N + N^i\mathcal{C}_{Ni} + \theta^{ij}\dot{\phi}_{ij} + L\mathcal{C}_L + L^i\mathcal{C}_{Li} - 2m^4N\sqrt{\gamma}\sum_{n=0}^{4}\beta_n e_n(S)\Big], \quad (1.26)$$

where L, L^i and ϕ_{ij} are the lapse, shift and spatial matrix of $f_{\mu\nu}$ and θ^{ij} the canonical momentum of ϕ_{ij}. As expected, the lapses and shifts do not appear linearly in the interaction terms. However, as the interaction terms do only depend on the shifts in the form of $N^i - L^i$ it is possible to make the field redefinition [24] [8] [27]

$$N^i - L^i = Ln^i + ND^i_k n^j, \quad (1.27)$$

with which we are then able to express one set of the shifts in terms of the new shift variable n^i. The specific shape of the matrix D is not relevant here, but it is chosen to make sure the action is linear in both lapses, for more details see [4]. Using this redefinition, the action then becomes

$$\mathcal{S}_{HR} = \int \mathrm{d}^4x \Big(\pi^{ij}\dot{\gamma}_{ij} + \theta^{ij}\dot{\phi}_{ij} + N\mathcal{C}_N + L\mathcal{C}_M + L^i\mathcal{C}_{Li}\Big). \quad (1.28)$$

We arrived at the same shape as we did for GR before and are able to count the DoF[2]. A theory containing a massless and a massive spin-2 field should in total have $4 + 10 = 14$ DoF in phase space. Currently we have a total of 24 DoF, with 6 each from all the dynamical quantities, i.e. π^{ij}, θ^{ij}, γ_{ij} and ϕ_{ij}. The constraints $\mathcal{C}_N = 0$, $\mathcal{C}_L = 0$ and $\mathcal{C}_{Li} = 0$ remove 5 DoF, while general coordinate transformations can be used to eliminate an additional 4 DoF. This leaves one additional constraint, which remains to be found. After the field redefinition, the Hamiltonian of (1.28) can be brought into the form

$$H = \int \mathrm{d}^4x(\mathcal{H}_0 - N\mathcal{C}),$$

[2]Note that the constraints in (1.28) differ from (1.26) due to the redefinition (1.27)

where $\mathcal{H}_0$ is the entire Hamiltonian that does not depend on N. Since every constraint needs to be conserved in time, we inspect the Poisson bracket of $\mathcal{C}$ and $\mathcal{H}$. The result is

$$\begin{aligned}\dot{\mathcal{C}}(x) &= \{\mathcal{C}(x), H\} \\ &= \int \mathrm{d}^4y(\{\mathcal{C}(x), \mathcal{H}_0(y)\} - N(y)\{\mathcal{C}(x), \mathcal{C}(y)\}),\end{aligned}$$

where $\mathcal{C}$ and $\mathcal{H}_0$ are independent of N. This means, that if $\{\mathcal{C}(x), \mathcal{C}(y)\}$ vanishes, $\dot{\mathcal{C}}$ will be independent of N and therefore yield a secondary constraint. This secondary constraint then removes the last DoF. There had been objections against bimetric theory, claiming the bracket could be non-zero [28]. It has been proven in [9] that the bracket vanishes weakly and that the secondary constraint exists, making bimetric gravity a ghost free theory.

2 Gravity as gauge theory

This chapter will give the reader an overview of gauge theories in the context of spin-2 theories, e.g. general relativity. We will start out by giving an introduction into gauge theories. Moving on, we will look at the theory by MacDowell and Mansouri [5], which is a gauge theoretic formulation of GR based on the $SO(1,4)$ group. Finally, we will inspect an attempt to formulate bimetric theory as a gauge theory of the $SO(1,5)$ group. A lot of the mathematics is expressed in the language of differential forms, a quick primer of which can be found in Appendix 1.2.

2.1 Introduction

In chapter 1 we learned that it is possible to express GR and other theories describing gravitational phenomena in form of a spin-2 field theory. This is motivated by our current day understanding and mathematical formulation of the SM, where all particles are represented as fields of certain spin. A crucial part of the SM are *local gauge symmetries*, which ensure the renormalizability and unitarity of its theories. As we inspected GR, massive gravity and bimetric gravity, the only gauge symmetries admitted were the usual diffeomorphisms. The natural question to ask here is, whether there is a way to formulate GR or similar theories to admit further gauge symmetries and therefore taking one more step towards the unification of theories of gravity and the SM.
In gauge theories of gravity, we usually associate a vector space T_xM with every point x on a manifold M. The *gauge* is then just a basis of T_xM, where the choice of a specific gauge is usually called *gauge fixing* (remember (1.2)). A change of basis is called *gauge transformation* and is an element of a *gauge group*. The idea behind a gauge theory is to construct a theory, which stays invariant under all transformations contained within this group. The *generators* of the gauge group are a basis of the Lie algebra associated with the gauge group. The Lie algebra is the tangent space of the gauge group at it's identity. In other words, a generator of a gauge group can be interpreted as a vector pointing in the direction of the gauge transformation. For

example, consider a transformation group which describes a transformation along a spiral in one direction with constant radius. This group would consists of two generators. One generator would describe the translation along the given axis and the second generator would describe the rotation around said axis.
A gauge theory contains a gauge field for each such generator. Gauge fields gives us a way to define the *covariant derivative*, which prescribes how a vector in T_xM transforms as it is transported to another tangent space T_yM. The gauge fields connect the components of the vector with the generators and determine how strong their influence on the transformation of those components is.

2.2 MacDowell Mansouri gravity

MacDowell and Mansouri [5] were able to construct a gauge theory of the $SO(1,4)$ group, which results in the Palatini action (1.12). With A being the associated $SO(1,4)$-connection, or gauge field. A allows a splitting into an $SO(1,3)$-connection ω_μ^{AB} and a *coframe field*[1] e_μ^A, such that we can express the gauge field A

$$A_\mu = \begin{pmatrix} \omega_\mu & \frac{1}{l}e_\mu \\ -\frac{1}{l}e_\mu & 0 \end{pmatrix}, \tag{2.1}$$

where l is a constant with unit length, later to be related with the cosmological constant. This can be explained by the fact that there exists a local decomposition of the Lie algebra as vector spaces, namely

$$\mathfrak{so}(1,4) \cong \mathfrak{so}(1,3) \oplus \mathbb{R}^{(1,3)}.$$

In [5] this breaking down of the $SO(1,4)$ symmetry has been motivated by the fact that the theory needs to be invariant under Lorentz transformations, which are part of the $SO(1,3) \subset SO(1,4)$ group. An interesting feature of the splitting (2.1) is that it continues into the gauge curvature $F_A = \mathrm{d}_A A + A \wedge A$. The curvature[2] splits into the $\mathfrak{so}(1,3)$-valued 2-form $F_\omega = \mathcal{D}_\omega\omega - \frac{1}{l^2}e \wedge e$ and the $\mathbb{R}^{1,3}$-valued 2-form $F_e = \mathcal{D}_\omega e$, where $\mathcal{D}_\omega$ is the covariant derivative $\mathcal{D}_\omega := \mathrm{d} + \omega\wedge$. Note that $R = \mathcal{D}_\omega\omega$ is the

[1] A coframe field is a set of covectors which form a basis to the cotangent bundle.
[2] Sometimes also called *field strength*

curvature defined for the spin connection in section 1.2.4 [3].
After the splitting we are able to construct an action from the curvature. In their paper [5], MacDowell and Mansouri argue that the most general action one can construct is

$$\mathcal{S}_{MM} = \int \mathrm{Tr}(F_\omega \wedge F_\omega) = -\frac{1}{4}\int \mathrm{d}^4x F^{AB}_{(\omega)\mu\nu} F^{CD}_{(\omega)\rho\sigma} Q_{ABCD}\epsilon^{\mu\nu\rho\sigma}, \tag{2.2}$$

where tjhe trace is taken with respect to the Lorentz indices and the Q_{ABCD} is a constant which is symmetric under the exchange $\{AB\} \leftrightarrow \{CD\}$ and antisymmetric for $A \leftrightarrow B$ and $C \leftrightarrow D$[4]. The reason for the above construction is simple, but with major consequences. There is no metric on the manifold M, because remember, the metric in GR is a field itself. Without a metric we don't have a consistent way to define the raising and lowering of indices (as it was done in (1.1)) or a measure to make the integral covariant. The action (2.2) is the most general covariant action one can construct without a metric.
The action (2.2) can be related to the traditional vielbein action of GR with the help of the Palatini action (1.12) by adding the term $\int \mathrm{d}^4x(\mathcal{D}_\omega\omega \wedge \mathcal{D}_\omega\omega)$, which vanishes under variations, to S_{MM}. The Palatini action in the language of differential forms looks like

$$S_P = \int \mathrm{Tr}[e \wedge e \wedge R - \Lambda e \wedge e \wedge (e \wedge e)].$$

Using the expression of $F_\omega = R - \frac{1}{l^2}e \wedge e$, we can express $\mathcal{S}_{MM}$ as

$$\begin{aligned}\mathcal{S}_{MM} &= \int \mathrm{Tr}\left[R \wedge R + \frac{1}{l^4}e \wedge e \wedge (e \wedge e) - \frac{2}{l^2}e \wedge e \wedge R\right] \\ &= \mathcal{S}_P + \int \mathrm{Tr}[R \wedge R],\end{aligned}$$

. The variation of the action (2.2) yields the two equations

[3]The term *curvature* might get confusing here, since we use it a lot. To clarify, a curvature can only be defined in tandem with a connection, so when we talk about the ω curvature we mean $R = \mathcal{D}_\omega\omega$, while F_ω is the part of the curvature of A corresponding to ω.
[4]An example would be the Levi-Civita symbol ϵ_{ABCD}.

$$\mathcal{D}_\omega F_\omega = \mathcal{D}_\omega(e \wedge e) = 0 \quad \text{and} \tag{2.3}$$

$$e \wedge F_\omega = 0. \tag{2.4}$$

The first equation is just the expression of the torsion-free condition placed on the manifold on which GR is formulated and the second one is the coordinate-free expression of Einstein's equations in terms of the spin connection and the vielbein e. After fixing $l^2 = \frac{3}{\Lambda}$ the Einstein equation take the form

$$e \wedge R - \frac{\Lambda}{3} e \wedge e \wedge e = 0.$$

A last thing of note here is the fact that this entire process only works, if the $SO(1,4)$ symmetry is broken (A is split into ω and e) *before* the variation of the action. If we were to break the symmetry after the variation, we would not be able to recover the Einstein equations from the EoM. This can be easily shown by varying the action obtained from the total curvature F and imposing the splitting (2.1) afterwards [50].

2.3 The SO(1,5) action and equations of motion

Inspired by the approach outlined in section 2.2 and works by [29], Fawad Hassan and Luis Apolo constructed a theory that contains not one, but two coframe fields e and t [10]. They arrived at a bimetric action similar to the ones studied in [30, 31, 32] and were able to find an action which contains a Lagrangian density for a partial massless field. A partially massless field is a massive field, where the mass of the field saturates the *Higuchi bound* [33] at which an additional constraint arises, reducing the DoF of the field from 5 to 4 (therefore *partially* massless). This phenomenon is outside of the scope of this thesis, but it has been inspected for bimetric theory in several publications [30, 34, 35].
The gauge theory is constructed on the $SO(1,3) \times SO(2) \subset SO(1,5)$ group, which is the global symmetry group of a massless and a partially massless spin-2 field [36]. The generators J_{AB} of the $\mathfrak{so}(1,5)$ algebra fulfil the commutation relations

$$[J_{AB}, J_{CD}] = \eta_{AD} J_{BC} + \eta_{BC} J_{AD} - \eta_{AC} J_{BD} - \eta_{BD} J_{AC},$$

where $A, B \in \{0, ..., 5\}$ and η_{AB} is the Minkowski metric in 6 dimensions. The breaking of the symmetry will be done in a way that the subalgebra $\mathfrak{so}(1,3) \oplus \mathfrak{so}(2)$ is obtained. After introducing the notation

$$P_a^{(1)} = J_{a4}, \qquad P_a^{(2)} = J_{a5}, \qquad D = J_{45},$$

where $a, b \in \{0, \ldots, 3\}$ are the indices of the $\mathfrak{so}(1,3)$ subalgebra, the non-vanishing commutators of the algebra take on the form

$$[J_{ab}, J_{cd}] = \eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \tag{2.5}$$

$$[J_{ab}, P_c^{(i)}] = \eta_{bc}P_a^{(i)} - \eta_{ac}P_b^{(i)}, \tag{2.6}$$

$$[P_a^{(i)}, P_b^{(j)}] = \epsilon^{ij}\eta_{ab}D - \delta^{ij}J_{ab}, \tag{2.7}$$

$$[D, P_a^{(i)}] = \epsilon^{ij}P_{(j)a}. \tag{2.8}$$

In the above equations, we introduced the indices $\{i, j\} \in \{1, 2\}$, which label vectors of $SO(2)$, while a, b are interpreted as tangent space indices. Following the approach introduced in [29], the $SO(1,5)$ gauge field can be constructed from the generators, giving

$$\mathbb{A} = \frac{1}{2}\omega^{ab}J_{ab} + l^{-1}e^aP_a^{(1)} + l^{-1}t^aP_a^{(2)} + AD. \tag{2.9}$$

The fields in (2.9) are interpreted as Lorentz spin connection ω_μ^{ab}, two vielbeins e_μ^a and t_μ^a and a gauge vector field A_μ. The fields are one-forms, which is why we write $e^a = e_\mu^a \mathrm{d}x^\mu$. To gain insight about the transformation behaviour of the fields, we parametrise an infinitesimal $SO(1,5)$ transformation in the form

$$\lambda = \frac{1}{2}\Lambda^{ab}J_{ab} + \chi^aP_a^{(1)} + \zeta^aP_a^{(1)} + \chi D,$$

where Λ^{ab} parametrises a Lorentz transformation and χD describe an $SO(2)$ transformation [5] After applying the infinitesimal gauge transformation of the form $\delta_\lambda \mathbb{A} =$

[5] $\chi^aP_a^{(1)} + \zeta^aP_a^{(1)}$ describes the 'left-over' part of $SO(1,5)$, which will be broken in the construction of the action later on.

$\mathrm{d}\lambda + [\mathbb{A}, \lambda]$, we find that the vielbeins transform under $SO(2)$ transformations as a vector, namely as

$$\delta_\xi \begin{pmatrix} e^a_\mu \\ t^a_\mu \end{pmatrix} = \begin{pmatrix} 0 & -\xi \\ \xi & 0 \end{pmatrix} \begin{pmatrix} e^a_\mu \\ t^a_\mu \end{pmatrix}.$$

Under Lorentz transformations they transforms as

$$\delta_\Lambda e^a = -\Lambda^a{}_b e^b \qquad \text{and} \qquad \delta_\Lambda t^a = -\Lambda^a{}_b t^b.$$

The homogeneous transformation under Lorentz transformation motivates the identification of e^a_μ and t^a_μ as vielbeins and therefore leads naturally to the definition of two metrics

$$g_{\mu\nu} = e^a_\mu e_{a\nu} \qquad \text{and} \qquad f_{\mu\nu} = t^a_\mu t_{a\nu}, \tag{2.10}$$

which are invariant under local Lorentz transformations. Knowing the transformation behaviour under $SO(2)$ transformations, a $SO(1,3) \times SO(2)$ invariant metric can be defined:

$$G_{\mu\nu} \coloneqq g_{\mu\nu} + f_{\mu\nu}. \tag{2.11}$$

The curvatures $\mathbb{F} = \mathrm{d}\mathbb{A} + \mathbb{A} \wedge \mathbb{A}$, associated with the different fields, are

$$\begin{aligned}
\mathbb{F}^{ab}_J &= \frac{1}{2}(R^{ab} - l^{-2} e^a \wedge e^b - l^{-2} t^a \wedge t^b), \\
\mathbb{F}^a_{P(1)} &= l^{-1}(\mathcal{D}_\omega e^a + A \wedge t^a), \\
\mathbb{F}^a_{P(2)} &= l^{-1}(\mathcal{D}_\omega t^a - A \wedge e^a) \quad \text{and} \\
\mathbb{F}_D &= \mathrm{d}A + l^{-2} t^a \wedge e_a,
\end{aligned}$$

where $R^{ab} = \mathrm{d}\omega + \omega^a_c \wedge \omega^{cb}$ is the Riemann curvature and $\mathcal{D}_\omega = \mathrm{d} + \omega^a_c \wedge$ is the covariant derivative with respect to the spin connection ω. Now that all the ingredients have been introduced, the action can be constructed. The $SO(1,3) \times SO(2)$-invariant action from [10] is of the form

$$\mathcal{S} = \int F_J^{ab} \wedge F_J^{cd} \epsilon_{abcd} - \sigma^2 \int F_D \wedge \star F_D, \tag{2.12}$$

where the Hodge star $\star$ is defined with regard to the metric $G_{\mu\nu}$. Notable here is that by only considering the curvatures $\mathbb{F}_J$ and $\mathbb{F}_D$, which are associated to the generators of the $\mathfrak{so}(1,3)$ and $\mathfrak{so}(2)$ algebras respectively, the symmetry of the action gets broken from $SO(1,5)$ down to $SO(1,3) \times SO(2)$.
The EoM, obtained through variation of the action are then

$$\epsilon_{abcd}(e^b \wedge F_{P(1)}^{cd} + t^b \wedge F_{P(2)}^{cd}) = 0, \tag{2.13}$$
$$\sigma^2 t_a \wedge F_D + \epsilon_{abcd} e^b \wedge F_J^{cd} = 0, \tag{2.14}$$
$$\sigma^2 e_a \wedge F_D - \epsilon_{abcd} t^b \wedge F_J^{cd} = 0, \tag{2.15}$$
$$\mathrm{d}F_D = 0. \tag{2.16}$$

It is argued in [10], that the spin connection should be imaginary, i.e. of the form $\omega_\mu^{ab} = \tau_\mu^{ab} + i\sigma_\mu^{ab}$, because the EoM (2.13) can not be the correct EoM for two reasons. For this argument, we will talk about the EoM for ω

$$\epsilon_{abcd}(e^b \wedge F_{P(1)}^{cd} + t^b \wedge F_{P(2)}^{cd}) = e_a \wedge \mathcal{D}_\omega e_b + t_a \wedge \mathcal{D}_\omega t_b = 0. \tag{2.17}$$

First, the EoM above is $SO(2)$ invariant, but independent of A. However, it is argued in [34, 37, 38], there cannot be any PM interactions above cubic order present in the theory.
The other argument is that, following [29], the action is not interpreted as a first order action[6], but as a second order action, where the spin connection is constrained by

$$\mathbb{F}_K^a = 0,$$

where K is some $SO(1,5)$ generator in the complement of $SO(1,3) \times SO(2)$. It can be argued that this is also true for the construction of MacDowell and Mansouri, where there is only one generator in the complement of $SO(1,3)$, which manifests as the torsionless condition (2.4). In our case, K could be e, t or any linear combination

[6] An action linear in derivatives

thereof. The issue is, that for all choices, the only $SO(2)$ invariant solution for ω is $\omega^{ab} = 0$.
If we take the spin connection to be imaginary though, there is such an $SO(2)$-invariant constraint which we can define by using the complex vielbein

$$\psi^a = e^a + it^a. \tag{2.18}$$

The constraint is of the form

$$\mathbb{F}^a_H = \mathbb{F}^a_{P(1)} + i\mathbb{F}^a_{P(2)} = l^{-1}(\mathcal{D}_\omega - iA) \wedge \psi^a = 0, \tag{2.19}$$

where $\mathbb{F}^a_H$ is the curvature associated with ψ^a. As a summary, the requirement for ω to be complex comes from the requirements of it being $SO(2)$-invariant and satisfying non-trivial constraints. With the constraint (2.19) the spin connection is given by

$$\omega^{ab}_\mu = \psi^{\gamma a} D_{[\mu}\psi^b_{\gamma]} - \psi^{\gamma b} D_{[\mu}\psi^a_{\gamma]} - \psi^{\rho a}\psi^{\sigma b}\psi_{c\mu} D_{[\rho}\psi^c_{\sigma]}, \tag{2.20}$$

where $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative with respect to A_μ. | Having a complex ω requires us to write the action as

$$\mathcal{S} = \int F^{ab}_J \wedge F^{cd}_J \epsilon_{abcd} - \sigma^2 \int F_D \wedge \star F_D. \tag{2.21}$$

An additional constraint is needed, which is necessary to make sure that bimetric theories are equivalent in their vielbein and metric formulation [22]. This constraint is

$$e^a_\mu t_{a\nu} - t^a_\mu e_{a\nu} = 0. \tag{2.22}$$

This constraint is necessary to remove half of the antisymmetric parts of the vielbeins and to make sure no antisymmetric rank-2 tensor would propagate in the metric formulation of the theory (since there is also no such tensor in GR). The constraint (2.22) also leads to the absence of vielbeins in the $\mathbb{F}_D$ term of the action (2.21). The action with the complex spin connection and the vielbein (2.18) takes on the form

$$\mathcal{S} = -\frac{m_P^2}{2}\int \epsilon_{abcd}\left[\psi^a \wedge \psi^{*b} \wedge \left[\mathrm{Re}R^{cd} - \frac{1}{2l^2}\psi^c \wedge \psi^{*d}\right] - l^2\frac{\sigma^2}{2}\int F \wedge \star F\right], \quad (2.23)$$

where $F = \mathrm{d}A$ and we ignored the topological term $\int R^{ab} \wedge R^{cd}\epsilon_{abcd}$. Note that, instead of e^a and t^a, we now treat ψ^a and ψ^{*a} as the two independent vielbeins. With help of the symmetrization constraint (2.22), one can then find an expression for

$$S^\mu_\nu = (\sqrt{g^{-1}f})^\mu_\nu = e^\mu_a t^a_\nu$$

and write the above action as

$$\begin{aligned}\mathcal{S} = m_P^2 \int \mathrm{d}^4x\sqrt{g}e^\mu_a e^\nu_b \mathrm{Re}R^{ab}_{\mu\nu} + m_p^2 \int \mathrm{d}^4x\sqrt{f}t^\mu_a t^\nu_b \mathrm{Re}R^{ab}_{\mu\nu} \\ - 2m_p^2 l^2 \int \mathrm{d}^4x\sqrt{g}\sum_{n=0}^{4}\beta_n e_n(S) - m_P^2 l^2\frac{\sigma}{2}\int \mathrm{d}^4x\sqrt{G}G^{\mu\alpha}G^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}, \quad (2.24)\end{aligned}$$

where we used the expressions for the elementary symmetric polynomials $e_n(S)$ derived in [22]. The β_n are given by

$$\beta_0 = 3, \qquad \beta_2 = 1, \qquad \beta_4 = 3 \quad \text{and } \beta_1 = \beta_3 = 0.$$

The rewritten action (2.24) is similar to the action of the bimetric theoy(1.23) with differing kinetic terms. Under the rescaling $t \to \alpha t$ and in the limit $\sigma \to \infty$ and $\alpha \to 0$ or ∞ with $\alpha^2 l^{-2}$ finite, one can decouple the two metrics and recover the Einstein-Hilbert action of GR with a positive cosmological constant. In addition, the limits $\alpha \to 0$ or ∞ will turn the spin connection into the real spin connection of GR for either of the vielbeins.

Similarly to bimetric theory, the action (2.24) allows proportional background solutions, i.e. where the vielbeins are connected by an arbitrary constant c, with the relations

$$e^a = \bar{e}^a \qquad t^a = c\bar{e}^a \quad \text{and } A = 0. \tag{2.25}$$

The resulting EoM reduces to

$$\epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}\bar{e}^b_\nu\left(R^{cd}_{\alpha\beta} - 2l^{-2}(1 + c^2\bar{e}^c_\alpha\bar{e}^d_\beta\right) = 0, \tag{2.26}$$

where $R^{ab}_{\alpha\beta}$ is the Riemann curvature for the real spin connection given by

$$\omega^{ab}_\mu = \bar{e}^{\gamma a}\partial_{[\mu}\bar{e}^b_{\gamma]} - \bar{e}^{\gamma b}\partial_{[\mu}\bar{e}^a_{\gamma]} - \bar{e}^{\rho a}\bar{e}^{\sigma b}\bar{e}_{c\mu}\partial_{[\rho}\bar{e}^c_{\sigma]}.$$

The EoM (2.26) is equivalent to Einstein's equations (1.6) with a cosmological constant given by

$$\Lambda = 3l^{-2}(1 + c^2).$$

As shown above, the theory admits de Sitter backgrounds and yields the correct EoM for GR in such a case. However, as we noted in section 1.4, the action of bimetric gravity does contain enough constraints to eliminate unwanted ghosts from the theory, but this has not yet been proven for the $SO(1,5)$ theory with the kinetic terms of (2.24). It could be that a theory with kinetic terms of the form above propagate additional DoF which cannot be removed. Recently, though, similar terms have been inspected [39, 40, 41], resulting in the discovery of additional constraints that remove ghosts beyond linear order [42, 43].

3 The Hamiltonian formalism

This chapter will give the reader a thorough introduction into the Hamiltonian formalism. The concepts introduced here transfer in most parts to the DDW formalism and will help bridge the gap between the unknown and the known for readers not yet familiar with the main topic of this thesis.
In the following we will introduce core concepts like the *Hamiltonian vector fields* and *symplectic forms*. We will also deal with the *constrained Hamiltonian*, which will come in handy later on, when we work with the constrained DDW Hamiltonian in the context of BT. This section is, unless specified otherwise, largely based on the chapter *The Hamilton method* in the brilliant lectures of P. Dirac [11] and chapter 3 of [44].

3.1 The Hamiltonian

Field theories are generally expressed in terms of an action S, which is the functional

$$S = \int \mathrm{d}^4x\mathcal{L} = \int \mathrm{d}tL, \tag{3.1}$$

where $\mathcal{L}$ is called the Lagrangian density and is defined such that its integral over space becomes the Lagrangian L of the theory. The Lagrangian is a function of the DoF, the fields q_i, their time derivatives $\dot{q}_i$ and the coordinates x^μ. In field theory, the fields are themselves functions of x^μ. The expression (3.1) already contains all the information about a theory. Varying the action and applying the *principle of least action*, i.e. the principle that physical systems follow dynamics which minimize (3.1), we get the *Euler - Lagrange equations*

$$\frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}. \tag{3.2}$$

Note that in the above formulation we used partial derivatives. In the case, where the q_i depend on x^μ, one would replace all partial derivatives in the above and the following equations with the *functional derivative*

$$\frac{\delta L}{\delta q_i} \coloneqq \frac{\delta}{\delta q_i} \int \mathrm{d}^3 x \mathcal{L} = \int \mathrm{d}^3 \left(\frac{\partial \mathcal{L}}{\partial q_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \right).$$

To move to the Hamiltonian formalism, we define the canonical momenta

$$p^i = \frac{\partial L}{\partial \dot{q}_i}, \tag{3.3}$$

with the help of which we can now introduce the *Hamiltonian*

$$H \coloneqq p^i \dot{q}_i - L. \tag{3.4}$$

The Hamiltonian is a function of q and p, though not of $\dot{q}$. We can see this by varying H, which gives us $\delta(p^i \dot{q}_i - L) = \delta p^i \dot{q}_i - (\frac{\partial L}{\partial q_i})\delta q_i$, i.e. all information about the dynamics of H is contained in p and q.

Generally, to solve the Hamiltonian equations we try to express the $\dot{q}$ in terms of the Hamiltonian variables. From the above variation we also obtain the just mentioned *Hamiltonian equations*

$$\frac{\partial H}{\partial q_i} = -\dot{p}^i \quad \text{and} \tag{3.5}$$

$$\frac{\partial H}{\partial p^i} = \dot{q}_i. \tag{3.6}$$

These equations are another way for us to compute the equations of motion for our system. But here we come to a very important point. The observant reader might already have noticed that the Hamiltonian (3.4) is not necessarily unique. We can add special functions $C(q_i, p^i) = 0$ to it, called *constraints*, which will not change the Hamiltonian itself. Why is this important? In many theories there exist constraints, which restrict the degrees of freedom this theory has. The degrees of freedom that are left give a physicist a way to gauge whether a theory is physical or not. Introduced by Dirac in [11], there exists a straight-forward way to find and compute all constraints a Hamiltonian theory contains. Expanding the time derivative in (3.2) we get

$$W_{ij}\ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} - \frac{\partial L}{\partial q^i} = 0, \tag{3.7}$$

where $W_{ij} := \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = \frac{\partial p_j}{\partial \dot{q}^i}$. If W_ij is invertible, then the $\ddot{q}^i$ are uniquely determined by a function of q^i and p^j. However, is W_{ij} not invertible, i.e. $\det(W_{ij}) = 0$, then there exist constraints $\phi^A(q^i, p_j) = 0$ restricting the amount of independent variables in the Hamiltonian or in other words, restring the amount of DoF in phase space. These constraints are called *primary constraints*. We can now extend not only the Hamiltonian but also the Hamiltonian equations. The Hamiltonian containing all primary constraints with coefficients λ_i is called the *total Hamiltonian*

$$H_{\text{tot}} = H - \lambda_A \phi^A \tag{3.8}$$

and its equations become

$$\dot{q}_i = \frac{\partial H}{\partial p^i} - \lambda_A \frac{\partial \phi^A}{\partial p^i} = \frac{\partial (H - \lambda_A \phi^A)}{\partial p^i} - \frac{\partial \lambda_A}{\partial p_i}\phi^A \approx \frac{\partial H_{\text{tot}}}{\partial p^i} \quad \text{and} \tag{3.9}$$

$$\dot{p}^i = -\frac{\partial H}{\partial q_i} + \lambda_A \frac{\partial \phi^A}{\partial q_i} = -\frac{\partial (H - \lambda_A \phi^A)}{\partial q_i} - \frac{\partial \lambda_A}{\partial q_i}\phi^A \approx -\frac{\partial H_{\text{tot}}}{\partial q_i}. \tag{3.10}$$

The sign $\approx$ used in the last equality is called a *weak equality*, denoting that this equation holds only on the *constraint surface* defined by all constraints $\phi^A = 0$. This surface sits in the phase space spanned by all q_i and p^i and it's dimension is equal to the amount of degrees of freedom the theory possesses. In most cases, the primary constraints do not yield a full set of constraints. This means, that there are still constraints to be uncovered and for that Dirac developed a straightforward method [11]. But before we move further, we need to become familiar with an important mathematical tool. The classical *Poisson bracket* is defined as

$$\{f(q,p), g(q,p)\} := \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p^i}. \tag{3.11}$$

It gives us the possibility to write the Hamiltonian equations (3.9) in the form

$$\dot{q}_i = \{q_i, H_{tot}\}, \qquad \dot{p}^i = \{p^i, H_{tot}\}. \tag{3.12}$$

More generally, it is also true that for any function $F(q_i, p^i)$,

$$\dot{F} \approx \{F, H_{tot}\}. \tag{3.13}$$

We see that, given any function F of phase space, its time evolution can be completely described by the Poisson bracket between H_{tot} and itself. It is said H_{tot} *generates* the change in F.
The expression (3.11) represents the local form of the Poisson bracket and is obtained by choosing appropriate q_i and p^j. In a more general context the properties of the Poisson bracket are captured by the *Poisson tensor* $\mathcal{P}^{mn}$. The bracket then takes on the form

$$\{f, g\} = \mathcal{P}^{mn}(\partial_m f)(\partial_n g), \tag{3.14}$$

where ∂_m has indices for both q and p. The Poisson tensor is antisymmetric and satisfies

$$\epsilon_{mno}\mathcal{P}^{mn}\partial_p\mathcal{P}^{op} = 0,$$

which is a consequence of the Poisson bracket fulfilling the Jacobi identity.
Although invertibility is not a given with $\mathcal{P}^{mn}$, if it exists, we call the inverse Ω_{mn} the *symplectic form*. Unlike in Riemannian geometry, where we can simply use the metric lowering the indices to indicate the inverse of the metric, in Poisson geometry the inverse is $-\mathcal{P}_{mn}$, which is why we use the symbol Ω. Equipped with the symplectic form, we now possess another tool which enables us to express the Poisson bracket as

$$\Omega(X_f, X_g) = \Omega_{mn}\mathcal{P}^{mo}\mathcal{P}^{np}\partial_o f\partial_p g = -\{f, g\}. \tag{3.15}$$

From the symplectic form we can define a vector field X_f for an arbitrary function f called the *Hamiltonian vector field*, iff the interior product $X_f \lrcorner \Omega$ is an exact 1-form (for a short introduction into forms and the interior product $\lrcorner$, see Appendix 1.2). The interior product $\lrcorner$ is a contraction of a p-form and vector field, yielding a $p-1$-form. For a more thorough explanation see Appendix 1.3. The Hamiltonian vector field induces a *Hamiltonian flow* on the manifold it is defined on. This flow may be interpreted as the direction of change in phase space associated to f. The Hamiltonian H has a canonical vector field X_H associated with it. The symplectic form also relates the Hamiltonian with the Hamiltonian vector field in the way

$$X_H \lrcorner \Omega = \mathrm{d}H, \tag{3.16}$$

where d is the exterior derivation, the explanation of which can be also found in Appendix 1.2. Using the definition of the Poisson brackets introduced in (3.15), one can then compute the EoM for p^i and q^i. In [45] E. Cartan introduced an even more general formulation of the relation (3.16) using the Poincaré - Cartan (PC) form

$$\Theta = p_i \mathrm{d}q^i - H\mathrm{d}t, \tag{3.17}$$

showing that the EoM appear as solutions for curves of X_H annihilating $\mathrm{d}\Theta$, i.e

$$X_H \lrcorner \mathrm{d}\Theta = 0, \tag{3.18}$$

a more detailed explanation of this can be found in Appendix 1.4.

As mentioned before, the primary constraints ϕ^A might not be enough to give a full set of constraints for the theory. We can obtain further constraints by requiring $\dot{\phi}^A = 0$. From (3.13) we know that $\dot{\phi}^A = \{\phi^A, H_{tot}\}$, which gives us a set of A equations

$$\{\phi^A, H\} + \lambda_B\{\phi^A, \phi^B\} = 0.$$

Some of those equations might reduce to identically 0, meaning we can leave those be and concentrate on the equations, which take the form

$$\chi^B(q, p) = 0, \tag{3.19}$$

and represent another constraint on the theory. These are called *secondary constraints*. Since these constraints are also required to be conserved in time, we follow the same procedure as for the ϕ^A and get another system of B equations

$$\{\chi^B, H\} + \lambda_C\{\chi^B, \phi^C\} = 0.$$

We carry on like that until no new constraints come up and we have discovered all secondary constraints.

From here on, all constraints will be treated on equal footing and we drop the distinction between primary and secondary constraints. Instead, we differentiate the constraints by whether their Poisson bracket $\{\phi^A, \cdot\}$ vanishes weakly with all other constraints or not. If it does, the constraint is called *first-class*, otherwise it is a *second-class* constraint. If all constraints are second-class, we can solve for all multipliers λ_A and determine the dynamics of the theory uniquely. If first-class constraints are present, the dynamics will not be unique and there are *gauge symmetries* present.

The obvious question here is: What are gauge transformations? To answer that, first consider a Hamiltonian

$$H = H_0 + \lambda_C \mathcal{C}^C,$$

where all the second-class constraints have already been solved and incorporated into H_0 and only the first-class constraints $\mathcal{C}_C$ and their undetermined multipliers λ_C are left. For any function f and differing λ_{C_1} and λ_{C_2}, the infinitesimal time evolution of $f = f(t = 0)$ is

$$f_j(t) = t(\{f, H_0\} + \{f, \phi^i\}\lambda_{i_j}), \qquad j = 1, 2. \tag{3.20}$$

From (3.20) it is clear that $f_i(t) \neq f_j(t)$ for $t \neq 0$, but they are actually only two different mathematical representations of the same physical state, a necessary conclusion for a well-defined theory. The transformation $\delta_\lambda f = f_1(t) - f_2(t) = \{f, \lambda_i \phi^i\}$, with $\lambda_i = t(\lambda_{i_1} - \lambda_{i_2})$, which maps the mathematical representations into each other is called a gauge transformation.

3.2 General Relativity

This section gives some more insights into the Hamiltonian formulation in the context of a field theory, by applying it to GR. We will not go through the calculations in detail, but instead try to give a short and concise overview over the approach that is applied to field theories to both compute the EoM and count the propagating DoF.
Using the 3+1 formulation introduced in section 1.2.2, we express the metric of GR as

$$g = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ N_i & \gamma_{ij} \end{pmatrix}. \tag{3.21}$$

Expressing the Einstein-Hilbert action in terms of the lapse, shift and the spatial metric γ, we see that the action is independent of time derivatives of N and N_i, leaving us only with the canonical momentum of γ

$$\pi^{ij} := \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} = \frac{\sqrt{\gamma}}{2N}\Big[\Big(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl}\Big)\dot{\gamma}_k l - \nabla^i N^j - \nabla^j N^i 2\gamma^{ij}\nabla^k N_k\Big]. \tag{3.22}$$

The Einstein-Hilbert action then becomes

$$\mathcal{S} = \int \mathrm{d}^4x \big[\pi^{ij}\dot{\gamma}_{ij} - \mathcal{H}\big] = \int \mathrm{d}^4x \big[\pi^{ij}\dot{\gamma}_{ij} + N\mathcal{C} + N^i\mathcal{C}_i\big], \tag{3.23}$$

where

$$\mathcal{C} = \sqrt{\gamma}R(\gamma) + \frac{1}{\sqrt{\gamma}}\left(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}\right),$$
$$\mathcal{C}_i = 2\sqrt{\gamma}\nabla^j \pi_{ij}$$

are the constraints which follow naturally from the lapse and shift taking on the role of Lagrange multipliers, with $R(\gamma)$ being the Ricci scalar of γ. The EoM for π and γ, the specific form of which does not matter for this calculation, follow from the Hamiltonian equations. The EoM for γ is simply equivalent to the definition of

π^{ij}(3.22). The EoM of π together with the Hamiltonian equations obtained from N and N_i, namely

$$\mathcal{C} = 0 \qquad \text{and} \qquad \mathcal{C}_i = 0,$$

are equivalent to the vacuum equation $R^{\mu\nu} = 0$. Note that N and N_i are appearing as Lagrange multiplier, therefore their EoM become constraints on γ and π.
The next step is to compute the consistency conditions

$$\dot{\mathcal{C}} = \{\mathcal{C}, H\} = 0 \qquad \text{and} \qquad \dot{\mathcal{C}}_i = \{\mathcal{C}_i, H\} = 0.$$

From (3.23) we know that the Hamiltonian only depends on $\mathcal{C}$ and $\mathcal{C}_i$, which is why we only need to compute the brackets

$$\begin{aligned}
\{\mathcal{C}(y), \mathcal{C}(x)\} &\propto \mathcal{C}_i, \\
\{\mathcal{C}(y), \mathcal{C}_i(x)\} &\propto \mathcal{C}, \\
\{\mathcal{C}_i(y), \mathcal{C}_j(x)\} &\propto \mathcal{C}_k.
\end{aligned}$$

The above relations give us two pieces of important information. First, the constraints $\mathcal{C}$ and $\mathcal{C}_i$ are the only constraints and, secondly, they are both first-class, since their brackets vanish weakly. This knowledge equips us with everything we need to count the DoF the theory propagates.
We start with a total of 16 DoF, with 6 from each π^{ij} and γ_{ij}, 1 from N and N^i contributing 3 DoF. We can fix 4 DoF using the EoM we computed above, which leaves us with 12 DoF. Another 4 DoF can be eliminated through general coordinate transformations. The last 4 DoF are fixed by $\mathcal{C}$ and $\mathcal{C}_i$. In the end, we are left with 4 DoF, which is the correct amount for a massless field and its momentum.

4 De Donder-Weyl formalism

This section will introduce the DDW formalism and the corresponding constraint analysis [12]. The aim will be to give comprehensive, but high-level, insights into the topic and focus more on the tools necessary for application than the deeper mathematical workings of the formalism. Complex details will be put in the Appendix 3 and will be referenced where applicable. There will, however be some passages where we cannot omit those details in the main text.
We will start by giving an introduction to the formalism and showing an application to a system of interacting real scalar fields. Then we will introduce the analogue to the Poisson brackets in the DDW formalism and outline some major differences between the canonical formalism and the DDW approach. Following that we will outline the constraint analysis in the formalism.

4.1 Introduction

There exist several approaches to the formulation of Hamiltonian field theory. The first and more well-known approach is the one we introduced in chapter 3, which implies a certain space and time decomposition. One other approach, called the *polysymplectic* formalism, originates in the works from T. De Donder [1], H. Weyl [2] and C. Carathéodory [3] and is entirely space-time covariant. This means that, both space and time variables enter and propagate the evolution of the theory on equal footing[1].
The *De Donder-Weyl* formalism is one such approach, where the generalized coordinates ϕ^a are the field variables to which we define a set of canonically conjugate momenta

$$\pi_a^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}, \tag{4.1}$$

[1] *Evolution* in this context can mean both time-evolution or any other space-time development

where $\mathcal{L}$ is the Lagrangian density. The DDW Hamiltonian can then be constructed in the form [2]

$$H_{DW}(\phi^a, \pi_a^\mu) := \pi_a^\mu \partial_\mu \phi^a - \mathcal{L}(\phi^a, \pi_a^\mu). \tag{4.2}$$

The Euler Lagrange equations can be written in the form ([46, 47, 48])

$$\partial_\mu \pi_a^\mu = -\frac{\partial H_{DW}}{\partial \phi^a} \qquad \text{and} \quad \partial_\mu \phi^a = \frac{\partial H_{DW}}{\partial \pi_a^\mu} \tag{4.3}$$

and are called *(DDW) Hamiltonian equations* due to their similarity to the canonical Hamiltonian equations.

Instead of the phase space spanned by (ϕ_μ, π_μ) like the phase space of the canonical Hamiltonian formulation, we work with a phase space called *extended polymomentum phase space* spanned by

$$z^M := (x^\mu, \phi^a, \pi_a^\mu), \quad 1 \leq M \leq m + mn + n, \tag{4.4}$$

where $\mu \in \{1 \ldots m\}$ and $a \in \{1 \ldots n\}$.

The ability to compute a Hamiltonian in a covariant way and several other interesting features make this formulation seem like an attractive tool to inspect theories in the realm of GR. However, until lately there have only been a small number of publications that studied its application to relativistic field theories, some of them are [16], [14] and [15]. Recently there have been some interesting developments in this space. Some publications of note are inspecting the MM theory [50] mentioned in section 2.2 and BF gravity [51], a reformulation of GR which was introduced in [52]. A big part of the recent developments in this field were thanks to I. Kanatchikov ([12, 53, 54, 55, 56, 57, 58]) and most of the following chapter will be based on his work, unless otherwise specified.

4.2 An application, or a manual

Based on chap. 6.1 in [12] this section is supposed to give the reader an overview of the DDW formalism through a hands-on example. In this demonstration we will be

[2] Note that the DDW Hamiltonian is defined with the Lagrangian density $\mathcal{L}$ instead of the Lagrangian L.

looking at a system of interacting real scalar fields ϕ^a described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi_a + V(\phi), \tag{4.5}$$

with the interaction term $V(\phi)$. It is straight-forward to compute the polymomenta from the given Lagrangian, i.e.

$$\pi^\mu_a = \partial^\mu \phi_a.$$

The DDW Hamiltonian is then just

$$H_{DDW} = \pi^\mu_a \partial_\mu \phi^a - \mathcal{L} = \frac{1}{2}\pi^\mu_a \pi^a_\mu - V(\phi).$$

Having assembled all the necessary ingredients, we can compute the Hamiltonian equations, which become

$$\partial_\mu \pi^\mu_a = -\frac{\partial H_{DW}}{\partial \phi^a} = -\frac{1}{2}\partial_a(\pi^\mu_b \pi^b_\mu) + \partial_a V(\phi) \tag{4.6}$$

$$\partial_\mu \phi^a = \frac{1}{2}\partial^a_\mu(\pi^\nu_b \pi^b_\nu) = \pi^a_\mu. \tag{4.7}$$

The second equation gives us the known relation between ϕ and π. In the first equation, the first term on the right-hand side vanishes since $\partial_b \pi^a_\mu = \partial_\mu \partial^b \phi_a = \partial_\mu \delta^a_a = 0$, which leaves us with

$$\partial_\mu \pi^\mu_a = \Box \phi_a = \partial_a V(\phi).$$

The above equation is the well-known Klein-Gordon equation for interacting scalar fields.

4.3 Polysymplectic form and Hamiltonian multivector fields

4.3.1 Notation and definitions

We will follow the notation introduced by I. Kanatchikov in [12]. Extended polymomentum phase space can be viewed as the bundle $\Pi : \mathcal{Z} \to M$ over the space-time manifold M. The coordinates introduced in (4.4) are the local coordinates of $\mathcal{Z}$. The coordinates $z^v = (\phi^a, \pi^\mu_a)$ are referred to as *vertical*, while the space-time coordinates x_μ are referred to as *horizontal components*. We will follow through with this distinction for subspaces and other objects related to these coordinates. For an introduction into vertical/horizontal bundles and vectors, see Appendix 1.1.

Before we move on, let us introduce some important objects we will use in this chapter. A *vertical vector* is an element of $T^V\mathcal{Z} := \ker T\Pi$ with the coordinate expression

$$X^V := X^a \partial_a + X^\mu_a \partial^a_\mu = X^v \partial_v,$$

where ∂^a_μ denotes a derivative by π^μ_a and ∂_a is the derivative by ϕ^a. In the same way, a p-form $\overset{p}{F}$ is called *horizontal*, if it's coordinate representation can be expressed as

$$\overset{p}{F} := \frac{1}{p!} F_{\mu_1 \ldots \mu_p}(z) \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{4.8}$$

Note that this means that a contraction of a horizontal form with a vertical vector always vanishes. We call a p-form element of the subspace of $(p-q)$ horizontal forms Λ^p_q, if its contraction with any $(q+1)$ vector vanishes.

We call a p-multivector field $\overset{p}{X}$ *vertical*, if its inner product with any p-form vanishes. In the coordinates (4.4) it takes the form

$$\overset{p}{X}{}^V := \frac{1}{p!} \overset{p}{X}{}^{v_1 \ldots v_q \mu_{q+1} \ldots \mu_{p-1}} \partial_{v_1} \wedge \cdots \wedge \partial_{v_q} \wedge \partial_{\mu_{q+1}} \wedge \cdots \wedge \partial_{\mu_{p-1}}, \tag{4.9}$$

where the index v references the vertical coordinate z^v, which we defined above.[3] The *vertical exterior derivative* d^V of an arbitrary p-form $\Phi = \frac{1}{p!}\Phi_{M_1 \dots M_p} \mathrm{d}z^{M_1} \wedge \dots \wedge \mathrm{d}z^{M_p}$ is given by

$$\mathrm{d}^V \Phi = \frac{1}{p!} \partial_v \Phi_{M_1 \dots M_p} \mathrm{d}z^v \wedge \mathrm{d}z^{M_1} \wedge \dots \wedge \mathrm{d}z^{M_p}. \tag{4.10}$$

4.3.2 The polysymplectic form

Starting out from a Lagrangian density $\mathcal{L}(x_\mu, \phi^a, \pi_a^\mu)$, we are able to construct the polymomenta

$$\pi_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}, \tag{4.11}$$

and the DDW Hamiltonian

$$H_{DW}(x^\mu, \phi^a, \pi_a^\mu) = \pi_a^\mu \partial_\mu \phi^a - \mathcal{L}(x_\mu, \phi^a, \pi_a^\mu). \tag{4.12}$$

This is analogous approach is the way undergraduate students first learn to compute the traditional Hamiltonian and the resulting Hamiltonian equations from a given Lagrangian. However, there is another approach using the Poincaré (PC) form, which we introduced in (3.16) (or more detailed in Appendix 1.4). The expression $X_H \lrcorner \Omega = \mathrm{d}H$ from (3.16) can be restated in the DDW formalism as

$$\overset{n}{X}{}^V \lrcorner \Omega^V = (-1)^n \mathrm{d}^V H_{DW}, \tag{4.13}$$

where n is the spacetime dimension of the manifold M and the superscript V denotes the vertical components defined above. The *polysymplectic form* Ω^V is obtained from the PC form of the DDW formalism, which is known to be

$$\Theta_{DW} = \pi_a^\mu \wedge d\phi^a \wedge \partial_\mu \lrcorner V - H_{DW} V, \tag{4.14}$$

where V is the volume form given by $V := \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m$ (see [59, 60]). From here on we use $V_\mu := \partial_\mu \lrcorner V$. The exterior differential of Θ is the form

[3] The wedge product is defined for both vectors and covectors, see Appendix 1.2

$$\Omega_{DW} = \mathrm{d}\pi^\mu_a \wedge \mathrm{d}\phi^a \wedge \partial_\mu \lrcorner V - \mathrm{d}H_{DW} V. \tag{4.15}$$

As shown in section 1.4 the variational principle of least action can be reformulated in a way stating that the Hamiltonian vector fields of the allowed trajectory in phase space annihilate the differential 2-form $\Omega = -\mathrm{d}\Theta$ [4]. The vertical parts of Ω are obtained by first separating Θ into its vertical and horizontal part, yielding for the vertical part

$$\Theta^V = -\mathrm{d}\phi^a \wedge \pi^\mu_a \wedge V_\mu.$$

After that we take the vertical derivative from (4.10) of Θ^V, which gives us

$$\Omega^V := -\mathrm{d}\phi^a \wedge \mathrm{d}\pi^\mu_a \wedge V_\mu. \tag{4.16}$$

Ω^V is called the *polysymplectic form*. The *co-exterior product* of two forms is defined as[5]

$$\overset{p}{F} \bullet \overset{q}{F} := \star^{-1}\left(\star \overset{p}{F} \wedge \star \overset{q}{F}\right), \tag{4.17}$$

which is both graded commutative [6] and associative. The inverse Hodge star is defined as $\star^{-1} \star \omega^{(p)} = \omega^{(p)}$ for an arbitray p-form.

4.4 Poisson-Gerstenhaber brackets and equations of motion

An important distinction between the traditional Hamiltonian formalism and the DDW formalism is the fact that instead of vectors, we work with *multivectors* (see Appendix 1.2). For these multivectors a generalized Lie derivative for an arbitrary q-form Φ is introduced [12]

[4]The interested reader is also referred to [60] or [61].
[5]You can find a definition of $\star$ in section 1.2
[6]I.e it picks up a factor $(-1)^{pq}$ under commutation.

$$L_{\overset{p}{X}}\Phi := \overset{p}{X} \lrcorner \mathrm{d}^V\Phi - (-1)^p \mathrm{d}^V(\overset{p}{X} \lrcorner \Phi).$$

Starting from this derivative it is possible to define a generalized *Lie bracket* of multivector fields, which we will need to construct a Poisson-style bracket for the DDW formalism .
Given Ω it is possible to define *Hamiltonian multivector fields* $\overset{p}{X}_F$ for which there exist a horizontal $(n-p)$-form $\overset{n-p}{F}$, s.t.

$$\overset{p}{X}_F \lrcorner \Omega = \mathrm{d}^V \overset{n-p}{F}. \tag{4.18}$$

The Poisson-Gerstenhaber bracket between two Hamiltonian multivector fields can now be defined through

$$\left[\overset{p}{X}_1, \overset{q}{X}_2\right] \lrcorner \Omega = L_{\overset{p}{X}_1} \mathrm{d}^V \overset{s}{F}_2 \tag{4.19}$$

$$= (-1)^{n-r} \mathrm{d}^V (\overset{p}{X}_1 \lrcorner \mathrm{d}^V \overset{s}{F}_2) \tag{4.20}$$

$$:= -\mathrm{d}^V \{\overset{r}{F}_1, \overset{s}{F}_2\}, \tag{4.21}$$

where $r = n-p$, $s = n-q$ and $\{.,.\}$ is the *graded Poisson-Gerstenhaber bracket* (PG) [12] defined for the DDW formalism

$$\{\overset{r}{F}_1, \overset{s}{F}_2\} := (-1)^{n-r} (\overset{p}{X}_1 \lrcorner \mathrm{d}^V \overset{s}{F}_2). \tag{4.22}$$

The bracket has several special properties, among those is that it's degree is given by

$$\deg\{\overset{p}{F}_1, \overset{q}{F}_2\} = p + q + 1 - n, \tag{4.23}$$

such that the bracket only exists if $p + q \geq n - 1$. The bracket is also graded anticommutativ, meaning

$$\{\overset{p}{F}_1, \overset{q}{F}_2\} = (-1)^{n-p-1}(-1)^{n-q-1}\{\overset{q}{F}_2, \overset{p}{F}_1\} \tag{4.24}$$

and fulfils the graded Jacobi identity

$$(-1)^{n-p-1}(-1)^{n-r-1}\{\overset{p}{F}, \{\overset{q}{F}, \overset{r}{F}\} + (-1)^{n-p-1}(-1)^{n-q-1}\{\overset{q}{F}, \{\overset{r}{F}, \overset{p}{F}\} \\ + (-1)^{n-q-1}(-1)^{n-r-1}\{\overset{r}{F}, \{\overset{p}{F}, \overset{q}{F}\} = 0$$

It is called Poisson-Gerstenhaber bracket, since the bracket together with the co-exterior product forms a *Poisson-Gerstenhaber algebra* [13] with properties (4.24), (4.25) and

$$\{\overset{p}{F}, \overset{q}{F} \bullet \overset{r}{F}\} = \{\overset{p}{F}, \overset{q}{F}\} \bullet \overset{r}{F} + (-1)^{(n-p-1)(n-q)} \overset{q}{F} \bullet \{\overset{p}{F}, \overset{r}{F}\}. \tag{4.25}$$

The DDW Hamiltonian H_{DW} is a form of degree 1. Therefore, following from the degree counting (4.23), the only forms that give a non-vanishing bracket with H_{DW} are Hamiltonian $n-1$-forms. Let us first compute the bracket of a general Hamiltonian $n-1$-form

$$F := F^{\mu} V_{\mu},$$

with the associated Hamiltonian vector field

$$X_F := X^a \partial_a + X^{\mu}_a \partial^a_{\mu}.$$

From the equation

$$X_F \lrcorner \Omega = \mathrm{d}^V F$$

when written in its components follow the two conditions

$$X^a \delta^\mu_\nu = -\partial^a_\nu F^\mu \tag{4.26}$$
$$X^\mu_a = \partial_a F^\mu. \tag{4.27}$$

The conditions (4.26) and (4.27) constrain the choice of $n-1$-forms, which can be Hamiltonian. From the first condition one can deduce the most general form of a Hamiltonian $n-1$-form to be

$$F^\mu = -\pi^\mu_a X^a(\phi, x) + g^\mu(\phi, x),$$

where X^a and g^i are arbitrary functions of ϕ and x. As next step we introduce the *total co-exterior derivative* $\mathbf{d}$ of the form F by x_μ

$$\mathbf{d}\bullet \overset{p}{F} := \frac{1}{(n-p)!}(\partial_a F^\nu \partial_\mu \phi^a + \partial^a_\rho F^\nu \partial_\mu \pi^\rho_a + \partial_\mu F^\nu) \mathrm{d}x^\mu \bullet V_\nu. \tag{4.28}$$

Using the Hamiltonian equations (4.3) and (4.27) we obtain the relation

$$\mathbf{d} \bullet F = (-1)^n \{H_{DW}, F\} V + \mathrm{d}^{hor} \bullet F, \tag{4.29}$$

where $d^{hor} \bullet F \sim \partial_\mu F^\nu \mathrm{d}x^\mu \bullet V_\nu$ appears only for forms which have an explicit space-time dependence. The canonical equations (4.3) of the DDW formalism can now be expressed in terms of the brackets we defined above. First, we construct two new $n-1$ -forms $\pi_a := p^\mu_a V_\mu$ and $\phi^a_\mu := \phi^a V_\mu$ and get the two equations

$$\mathbf{d} \bullet \pi_a = \{H_{DW}, \pi_a\} = -\partial_a H_{DW}, \tag{4.30}$$
$$\mathbf{d} \bullet \phi^a_\mu = \{H_{DW}, \phi^a_\mu\} = \partial^a_\mu H_{DW}, \tag{4.31}$$

which is exactly the Poisson bracket formulation of the Hamiltonian field equations. It is interesting to note here, that this entire construction relies on Hamiltonian $n-1$-forms, since those are the only forms that have non-vanishing brackets with the Hamiltonian H_{DW}, which is a 1-form. Although there has been some exploration done into a formulation allowing forms of degrees other than $n-1$ [12][58], so far there has been no significant progress made in this direction. This is somewhat cumbersome as we see in the following sections, where we will encounter other forms from which we will need to construct $n-1$ forms before we are able to continue.

4.5 Constraints

Same as in the canonical Hamiltonian formalism, there may be constraints in the DDW formalism. These constraints also appear in the same way and take on the form

$$C_A(\phi^a, \pi^\mu_a) = 0.$$

Following the canonical approach, these constraints are also called *primary constraints* to emphasize their stark similarity to the primary constraints of the Hamiltonian formalism (see [13]).
This means, that the DDW Hamiltonian is not unique and is weakly equal to

$$\tilde{H}_{DW} = H_{DW} + u_A C_A \approx H_{DW},$$

where u_A are arbitrary coefficients. Continuing the analogy with the canonical formalism, we will classify the constraints using the PG bracket. For that reason, as mentioned in the previous section, we will need to constrain ourselves to constraints from which we are able to construct Hamiltonian $n-1$-forms

$$\mathcal{C}_m = C^\mu_m V_\mu,$$

which contract with 1-forms u^m that function as Lagrange multipliers to $u^m \bullet \mathcal{C}_m = u_A C_A$ (note that A in this case is not an index). A constraint is called *first-class*, if it's PG bracket with every other constraint vanishes weakly, i.e.

$$\{\mathcal{C}_m, \mathcal{C}_n\} \approx 0 \qquad \forall \mathcal{C}_n.$$

If this is not the case for at least one constraint, it is called *second-class*. As with the canonical Poisson bracket, the first-class property is preserved under the PG bracket (see [13]). It is also necessary, that the constraints $\mathcal{C}_m$ fulfil the *consistency condition* $\mathbf{d} \bullet \mathcal{C}_m = 0$ for the total co-exterior derivative (4.28). From this condition follow the relations

$$\mathbf{d} \bullet \mathcal{C}_m = \{\tilde{H}_{DW}, \mathcal{C}_m\} = \{H_{DW} . \mathcal{C}_m\} + u^n \bullet \{\mathcal{C}_n, \mathcal{C}_m\} \approx 0. \tag{4.32}$$

Following Dirac's terminology [11] the above relation can either impose a constraint on the u^n or reveal a new constraint on ϕ^a and π_a^μ called *secondary constraints*. For those constraints it is again necessary to check whether they fulfil the condition (4.32) and whether they yield an additional secondary constraint or constrain the u^n.

5 De Donder-Weyl Hamiltonian Formalism applied to SO(1,5) Gravity

In this section we will apply the DDW formalism to the $SO(1,5)$ theory of [10]. In section 5.1 we will start by inspecting how the construction of the action and therefore the breaking of the symmetry from $SO(1,5)$ down to $SO(1,3) \times SO(2)$ affects the EoM obtained from the formalism. To keep the equations simple we will use a real spin connection. In the original paper it was argued that the spin connection must be complex. We will inspect the complex action in the following section 5.2 through the lens of the DDW Hamiltonian and see if are able to obtain the same results as in [10].

5.1 The equations of motion for SO(1,5) actions with real spin connections

The $SO(1,5)$ theory from [10] is, like the theory of MacDowell and Mansouri [5], a strongly geometric theory, which offers itself to analysis with geometric tools like the polysymplectic formalism by de Donder and Weyl [1, 2]. We will apply the DDW formalism to the action of the $SO(1,5)$, compute its EoM and study it's behaviour under the symmetry breaking from $SO(1,5)$ down to $SO(1,3) \times SO(2)$. Unlike the Lagrangian approach, where we would obtain differing EoM, depending on whether we break the symmetry before or after the variation of the action, we expect this to not make a difference for the DDW approach. This chapter can also be seen as a practical example and continuation to chapter 4 and can give more insight into applications and advantages of the formalism. More detailed calculations can be found in Appendix 2.

Instead of taking only the projection of $SO(1,5)$ down to $SO(1,3) \times SO(2)$ into consideration when constructing the action as in [10] (see (2.21)), we inspect the full $SO(1,5)$ action

$$I = \int \mathrm{Tr}(F \wedge F) = \int (F^{AB} \wedge F^{CD}) Q_{ABCD}, \tag{5.1}$$

where F is the curvature $F = \mathrm{d}\mathbb{A} + [\mathbb{A}, \mathbb{A}]$ of the entire $SO(1,5)$ gauge field (2.9). In components, $\mathcal{L}$ takes the form

$$\mathcal{L} = -\frac{1}{4} F^{AB}_{\mu\nu} F^{CD}_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} Q_{ABCD}, \tag{5.2}$$

where we used the usual convention for spacetime indices (greek) and the group indices $A \in \{0, 1, \ldots, 5\}$. The factor Q_{ABCD} originates from the formulation of MM in [5], where they introduced a tensor, where every combination of AB (or CB respectively) is tied to one of the generators of the Lie group and which is symmetric under $AB \leftrightarrow CD$ and fulfils the equation

$$f^{GH}{}_{ABEF} Q_{GHCD} - f^{GH}{}_{CDAB} K_{GHEF} = 0,$$

where the constants $f^{EF}{}_{ABCD}$ are the structure constants of the gauge group ($SO(1,4)$ in the MM case). The constant in this theory was used to construct the most general action from the curvature and turned out to be the Levi-Civita symbol ϵ_{ijkl} in the MM theory[1]. The tensor Q_{ABCD} is defined as

$$Q_{ABCD} = \frac{1}{2}(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}) \tag{5.3}$$

and fulfils both of the aforementioned requirements. From this we can go on and compute the polymomenta

$$\Pi^{AB}_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathbb{A}^{AB}_\nu)} = -F^{CD}_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} Q_{ABCD}. \tag{5.4}$$

Remembering (4.2), we can compute the Hamiltonian for the gauge fields $\mathbb{A}^{AB}_\mu$, namely

[1] Note that the Levi-Civita symbol is actually a tensor *density*, which transforms like a tensor field but is weighted by a power W of the Jacobi determinant of the coordinate transformation.

$$H_{DW} = -\frac{1}{16}\epsilon_{\mu\nu\rho\sigma}Q^{ABCD}\Pi^{\mu\nu}_{AB}\Pi^{\rho\sigma}_{CD} - \frac{1}{2}\Pi^{\mu\nu}_{AB}\mathbb{A}^{A}_{[\mu K}\mathbb{A}^{KB}_{\nu]}. \tag{5.5}$$

The De Donder-Weyl equations then take on the form

$$\partial_\mu \mathbb{A}^{AB}_{\nu} = -\frac{1}{8}Q^{ABCD}\epsilon_{\mu\nu\rho\sigma}\Pi^{\rho\sigma}_{CD} - \frac{1}{2}\mathbb{A}^{A}_{[\mu J}\mathbb{A}^{JB}_{\nu]} \quad \text{and} \tag{5.6}$$

$$\partial_\mu \Pi^{\mu\nu}_{AB} = -\mathbb{A}^{K}_{\mu[A}\Pi^{\mu\nu}_{B]K}. \tag{5.7}$$

If we inspect the first of the above equations more closely, we see that both terms on the left hand side are antisymmetric under $\mu \leftrightarrow \nu$. This allows us to express $\partial_\mu \mathbb{A}^{AB}_{\nu} = 2\partial_{[\mu}\mathbb{A}^{AB}_{\nu]}$, which we will use from now on. Following the same procedure as in section 2.3 and $a \in \{0, 1, 2, 3\}$, we introduce the notation

$$\omega^{ab}_{\mu} := \mathbb{A}^{ab}_{\mu} \qquad e^{a}_{\mu} := \mathbb{A}^{a4}_{\mu}$$
$$t^{a}_{\mu} := \mathbb{A}^{a5}_{\mu} \qquad A_\mu := \mathbb{A}^{45}_{\mu},$$

and continue this notation for the polymomenta, giving

$$\pi^{\mu\nu}_{ab} := \Pi^{\mu\nu}_{ab} \qquad p^{(1)\mu\nu}_{a} := \Pi^{\mu\nu}_{a4}$$
$$p^{(2)\mu\nu}_{a} := \Pi^{\mu\nu}_{a5} \qquad \theta^{\mu\nu} := \Pi^{\mu\nu}_{45}.$$

Before we express the De Donder-Weyl equations in this notation, let's mention that the only non-zero constants Q^{ABCD} are Q^{abcd}, $Q^{a4b4} = Q^{a5b5} = \frac{1}{2}\eta^{ab}$, Q^{4545} and, since they are antisymmetric in the first and last two indices, Q^{5a5a}, Q^{4a4a} and Q^{5454}. As always we use lower case latin indices to indicate an index running from 1 to 3. In the following, we will compute the EoM explicitly for the ω-field for readability. The derivations of the EoM for the other fields can be found in Appendix 2.1.
The first line in (5.7) gives us

$$\partial_\mu \omega^{ab}_{\nu} = -\frac{1}{8}\epsilon_{\mu\nu\rho\sigma}Q^{abcd}\pi^{\rho\sigma}_{cd} - \frac{1}{2}(\omega^{a}_{[\mu c}\omega^{cb}_{\nu]} - t^{a}_{[\mu}t^{b}_{\nu]} - e^{a}_{[\mu}e^{b}_{\nu]}), \tag{5.8}$$

which we can solve for $\pi^{\mu\nu}_{ab}$. To accomplish that, we need to use the relations

$$\epsilon_{i_1 \dots i_n j_{n+1} \dots j_m} \epsilon^{i_1 \dots i_n k_{n+1} \dots k_m} = n!(m-n)! \delta^{k_{n+1}}_{[j_{n+1}} \dots \delta^{k_m}_{j_m]} \qquad \text{and}$$
$$Q_{ABCD} Q^{CDEF} = \delta^E_{[A} \delta^F_{B]},$$

where $\delta^{k_{n+1}}_{[j_{n+1}} \dots \delta^{k_m}_{j_m]}$ signifies a total antisymmetrisation of all $m-n$ Kronecker deltas. In our case this becomes $\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\delta\gamma} = 4\delta^{\delta}_{[\rho}\delta^{\gamma}_{\sigma]}$. Using the above relations, we can solve for $\pi^{\mu\nu}_{ab}$ and obtain

$$\pi^{\mu\nu}_{ab} = -\epsilon^{\mu\nu\rho\sigma} Q_{abcd} (\omega^c_{[\mu e} \omega^{ed}_{\nu]} - t^c_{[\mu} t^d_{\nu]} - e^c_{[\mu} e^d_{\nu]} + 2\partial_{[\mu} \omega^{cd}_{\nu]}). \tag{5.9}$$

Restricting (5.7) to the index pair $\{a, b\}$, i.e. to $\pi^{\mu\nu}_{ab}$, gives the relation

$$\partial_\mu \pi^{\mu\nu}_{ab} = -\omega^c_{\mu[a} \pi^{\mu\nu}_{b]} + e_{\mu[a} p^{(1)\mu\nu}_{b]} + t_{\mu[a} p^{(2)\mu\nu}_{b]}. \tag{5.10}$$

We can now substitute $\pi^{\mu\nu}_{ab}$ from (5.9) and the expressions for the other polymomenta, which we derived in Appendix 2, into the above equation. The polymomenta of the vielbeins and A are

$$p^{(1)\mu\nu}_a = -\epsilon^{\mu\nu\rho\sigma} \eta_{ab} \Big(2\partial_{[\rho} e^b_{\sigma]} + \omega^b_{[\rho c} e^c_{\sigma]} - t^b_{[\rho} A_{\sigma]} \Big), \tag{5.11}$$
$$p^{(2)\mu\nu}_a = -\epsilon^{\mu\nu\rho\sigma} \eta_{ab} \Big(2\partial_{[\rho} t^b_{\sigma]} + \omega^b_{[rhoc} t^c_{\sigma]} + e^b_{[\rho} A_{\sigma]} \Big), \qquad \text{and} \tag{5.12}$$
$$\theta^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \Big(e_{[\rho c} t^c_{\sigma]} - 2\partial_\rho A_\sigma \Big). \tag{5.13}$$

The EoM for ω takes on the form

$$\begin{aligned} 0 = \epsilon^{\mu\nu\rho\sigma} Q_{abcd} \Big[\partial_\mu \Big(\omega^c_{[\mu e} \omega^{ed}_{\nu]} - t^c_{[\mu} t^d_{\nu]} - e^c_{[\mu} e^d_{\nu]} + 2\partial_{[\mu} \omega^{cd}_{\nu]} \Big) \\ + \omega^c_{\mu e} \Big(\omega^d_{[\rho f} \omega^{fe}_{\sigma]} - t^d_{[\rho} t^e_{\sigma]} - e^d_{[\rho} e^e_{\sigma]} + 2\partial_{[\rho} \omega^{de}_{\sigma]} \Big) \Big] \\ - \epsilon^{\mu\nu\rho\sigma} \eta_{bc} \Big[e_{\mu a} \Big(2\partial_{[\rho} e^c_{\sigma]} + \omega^c_{[\rho d} e^d_{\sigma]} - t^c_{[\rho} A_{\sigma]} \Big) \\ + t_{\mu a} \Big(2\partial_{[\rho} t^c_{\sigma]} + \omega^c_{[\rho d} t^d_{\sigma]} + e^c_{[\rho} A_{\sigma]} \Big) \Big], \end{aligned} \tag{5.14}$$

which looks rather complicated. To simplify equation (5.14), we can use the components of the curvature $F_{\mathbb{A}} = \mathrm{d}\mathbb{A} + [\mathbb{A}, \mathbb{A}]$, namely

$$
\begin{aligned}
F^{ab}_{\mathbb{A}\mu\nu} &\coloneqq F^{ab}_{(\omega)\mu\nu} = \frac{1}{2}R^{ab}_{\mu\nu} - e^a_{[\mu}e^b_{\nu]} - t^a_{[\mu}t^b_{\nu]} = \epsilon^{\mu\nu\rho\sigma}\pi^{cd}_{\rho\sigma}Q_{abcd} \\
F^{a4}_{\mathbb{A}\mu\nu} &\coloneqq F^a_{P^{(1)}\mu\nu} = 2\partial_{[\mu}e^a_{\nu]} + \omega^a_{[\mu c}e^c_{\nu]} - t^a_{[\mu}A_{\nu]} = \eta^{ab}\epsilon_{\mu\nu\rho\sigma}q_b^{(1)\rho\sigma} \\
F^{a5}_{\mathbb{A}\mu\nu} &\coloneqq F^a_{P^{(2)}\mu\nu} = 2\partial_{[\mu}t^a_{\nu]} + \omega^a_{[\mu c}t^c_{\nu]} + e^a_{[\mu}A_{\nu]} = \eta^{ab}\epsilon_{\mu\nu\rho\sigma}q_b^{(2)\rho\sigma} \\
F^{45}_{\mathbb{A}\mu\nu} &\coloneqq F_{(A)\mu\nu} = 2\partial_{[\mu}A_{\nu]} - e_{[\mu b}t^b_{\nu]},
\end{aligned}
$$

where $R^{ab}_{\mu\nu} \coloneqq \partial_{[\mu}\omega^{ab}_{\nu]} + 2\omega^a_{[\mu c}\omega^{cb}_{\nu]}$. Substituting the curvatures and expressing the EoM of ω in the language of differential forms, it takes on the shape

$$
Q_{abcd}\mathcal{D}_\omega F^{cd}_{(\omega)} = e_a \wedge F_{P^{(1)}b} + t_a \wedge F_{P^{(2)}b}, \tag{5.15}
$$

where $\mathcal{D}_\omega = \mathrm{d} + \omega\wedge$ is the usual covariant derivative with respect to ω. To derive this expression we used the fact that $\omega^c_{\mu[a}Q_{b]cde} = Q_{abcd}\omega^c_{\mu e}$. Now that we have computed the EoM, we can break the symmetry of the theory down to $SO(1,3) \times SO(2)$. This breaking of the $SO(1,5)$ symmetry is achieved by requiring $p^{(1)} = 0$ and $p^{(2)} = 0$, which is equivalent to saying $F^a_{P^{(1/2)}\mu\nu} = 0$ and gives us the equation of motion

$$
Q_{abcd}(e^c \wedge F^d_{P^{(1)}} + t^c \wedge F^d_{P^{(2)}}) = e_a \wedge \mathcal{D}_\omega e_b + t_a \wedge \mathcal{D}_\omega t_b = 0. \tag{5.16}
$$

This is a similar equation to (2.17), but differs from it by the factor Q_{abcd}. However, in this context Q_{abcd} gives exactly the same behaviour as ϵ_{abcd}, i.e. from its definition (5.3) we see that all combinations $c \neq d$ yield both a positive and a negative tensor, which cancel each other out, and the case $c = d$ vanishes by definition.
We are able to obtain the exact same equation as (2.17) by breaking the symmetry before the variation, like it is done in [10] and [5]. We accomplish this by constructing the action as

$$
\begin{aligned}
\mathcal{S} &= \int \mathrm{Tr}\big(F_{(\omega)} \wedge F_{(\omega)}\big) - \sigma^2 \int F_A \wedge \star F_A & (5.17) \\
&= \int F^{ab}_{(\omega)} \wedge F^{cd}_{(\omega)} Q_{abcd} - \sigma^2 \int F_A \wedge \star F_A. & (5.18)
\end{aligned}
$$

Doing this, the Hamiltonian takes on the form

$$H_{DW} = -\frac{1}{16}\epsilon_{\mu\nu\rho\sigma}(Q^{abcd}\pi^{\mu\nu}_{ab}\pi^{\rho\sigma}_{cd} + 2\theta^{\mu\nu}\theta^{\rho\sigma}) - \frac{1}{2}\Big[\pi^{\mu\nu}_{ab}(\omega^a_{[\mu c}\omega^{cb}_{\nu]} - e^a_{[\mu}e^b_{\nu]} - t^a_{[\mu}t^b_{\nu]}) - \theta_{\mu\nu}e_{[\mu a}t^b_{\nu]}\Big]. \tag{5.19}$$

The De Donder-Weyl equations for the ω-field are

$$\partial_\mu\omega^{ab}_\nu = \frac{H_{Dw}}{\partial\pi^{\mu\nu}_{ab}} = -\frac{1}{8}\epsilon_{\mu\nu\rho\sigma}Q^{abcd}\pi^{\rho\sigma}_{cd} - \frac{1}{2}\omega^a_{[\mu}\omega^{cb}_{\nu]} + \frac{1}{2}(e^a_{[\mu}e^b_{\nu]} + t^a_{[\mu}t^b_{\nu]}) \qquad \text{and} \tag{5.20}$$

$$\partial_\mu\pi^{\mu\nu}_{ab} = \frac{\partial H_{DW}}{\partial\omega^{ab}_{\mu\nu}} = -\pi^{\nu\sigma}_{ac}\omega^c_{\sigma b}. \tag{5.21}$$

Solving for π in the first line and then inserting the expression into the second equation in the same way we did before, we get the equation

$$e_a \wedge \mathcal{D}_\omega e_b + t_a \wedge \mathcal{D}_\omega t_b = 0$$

For the other fields, we get the equations

$$\begin{aligned}\sigma^2 t^a \wedge F_A + e^b \wedge F_{(\omega)ab} &= 0\\ \sigma^2 e^a \wedge F_A - t^b \wedge F_{(\omega)ab} &= 0\\ \mathrm{d}F_A &= 0.\end{aligned}$$

The above expressions are the same as in (5.16), showing that in the DDW formalism, the timing of the symmetry breaking has no influence on the result. If, instead of using Q_{abcd} in (5.18) we were to construct the action with ϵ_{abcd}, i.e. in exactly the same way as in 2.3 and [10], we get the EoM

$$\begin{aligned}\epsilon_{abcd}(e^c \wedge F^d_{P(1)} + t^c \wedge F^d_{P(2)}) &= 0,\\ \sigma^2 t^a \wedge F_A + \epsilon_{abcd}e^c \wedge F^d_{(\omega)} &= 0,\\ \sigma^2 e^a \wedge F_A - \epsilon_{abcd}t^c \wedge F^d_{(\omega)} &= 0,\\ \mathrm{d}F_A &= 0.\end{aligned} \tag{5.22}$$

Of course, we have slight deviations in both approaches, since we chose to use different constants, but they both fulfil the aforementioned requirements stated by MacDowell and Mansouri and yield therefore valid solutions.

5.2 The SO(1,5) equations of motion with a complex spin connection

Since the EoM (5.22) only give a solution, where $\omega = 0$, we also need to inspect the action containing the complex vielbein $\psi^a = e^a + it^a$ and complex spin connection $\omega = \tau + i\sigma$ that have been introduced in [10]. The complex vielbeins fulfill a $SO(2)$-invariant constraint, given by

$$\mathbb{F}^a_H = \mathbb{F}^a_{P(1)} + i\mathbb{F}^a_{P(2)} = (\mathrm{d}_\omega - iA) \wedge \psi^a = 0. \tag{5.23}$$

The action (5.18) then becomes

$$I = -\frac{1}{2}\int \epsilon_{abcd}(\psi^a \wedge \psi^{*b} \wedge (\mathrm{Re}R^{cd} - \psi^c \wedge \psi^{*b}) - \mathrm{Re}(R^{ab} \wedge R^{cd})) \tag{5.24}$$

$$- \frac{\sigma^2}{2}\int \mathrm{d}A \wedge \star\mathrm{d}A, \tag{5.25}$$

which is the action of equation 2.41 in [10] with the added topological term $\frac{1}{2}\int \mathrm{Re}(R^{ab} \wedge R^{cd})\epsilon_{abcd}$. Even though it drops out during the variation of the action and is usually omitted when writing down the action, it is necessary for the computation of the De Donder Weyl Hamiltonian. It is ignored in the original theory, since terms like this always vanish during variation of the action. To compute the polymomenta, we need to write the action in it's components

$$\begin{aligned} I = &-\frac{1}{8}\int \epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}\Big[2\psi^a_{[\mu}\psi^{*b}_{\nu]}(2\partial_{[\rho}\tau^{cd}_{\sigma]} + \tau^c_{[\rho e}\tau^{ed}_{\sigma]} - \sigma^c_{[\rho e}\sigma^{ed}_{\sigma]}) - 2\psi^c_{[\rho}\psi^{*b}_{\sigma]}) \\ &- (2\partial_{[\rho}\tau^{ab}_{\sigma]} + \tau^a_{[\rho e}\tau^{eb}_{\sigma]} - \sigma^a_{[\rho e}\sigma^{eb}_{\sigma]})(2\partial_{[\rho}\tau^{cd}_{\sigma]} + \tau^c_{[\rho e}\tau^{ed}_{\sigma]} - \sigma^c_{[\rho e}\sigma^{ed}_{\sigma]})\Big] \\ &- \frac{\sigma^2}{2}\int G^{\mu\alpha}G^{\nu\beta}\partial_{[\alpha}A_{\beta]}\partial_{[\mu}A_{\nu]}. \end{aligned}$$

The only non-vanishing polymomenta are

$$\mathrm{T}_{ab}^{\mu\nu} \coloneqq \frac{\partial \mathcal{L}}{\partial(\partial_\mu \tau_\nu^{ab})} = -\frac{1}{2}\epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}\left(-2\partial_{[\rho}\tau_{\sigma]}^{cd} - \tau_{[\rho e}^{c}\tau_{\sigma]}^{ed} + \sigma_{[\rho e}^{c}\sigma_{\sigma]}^{ed} + \frac{1}{2}\psi_{[\rho}^{c}\psi_{\sigma]}^{*d}\right) \tag{5.26}$$

$$\theta^{\mu\nu} \coloneqq \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\sigma^2 G^{\alpha\mu}G^{\beta\nu}\partial_{[\alpha}A_{\beta]}, \tag{5.27}$$

where the polymomenta associated with the complex vielbeins ψ and ψ^*, namely $\Psi_a^{\mu\nu}$ and $\Psi_a^{*\mu\nu}$ respectively, vanish because the vielbeins do not appear as derivatives in the action (remember the definition of the polymomentum $\pi_{AB}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathbb{A}_\nu^{AB})}$).
Having assembled all the ingredients, we can now write the De Donder-Weyl Hamiltonian as

$$\begin{aligned} H_{DW} = -\frac{1}{2}\Bigg(&\frac{1}{16}\epsilon^{abcd}\epsilon_{\mu\nu\rho\sigma}\mathrm{T}_{ab}^{\mu\nu}\mathrm{T}_{cd}^{\rho\sigma} + \frac{1}{\sigma^2}G_{\mu\alpha}G_{\nu\beta}\theta^{\alpha\beta}\theta^{\mu\nu} \\ &+ \mathrm{T}_{ab}^{\mu\nu}\left[\left(\tau_{[\rho e}^{c}\tau_{\sigma]}^{ed} - \sigma_{[\rho e}^{c}\sigma_{\sigma]}^{ed}\right) + \frac{1}{2}\psi_{[\rho}^{c}\psi_{\sigma]}^{*d}\right] \\ &- \frac{1}{4}\epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}\psi_{[\mu}^{a}\psi_{\nu]}^{*b}\psi_{[\rho}^{c}\psi_{\sigma]}^{*d}\Bigg). \end{aligned} \tag{5.28}$$

We can compute the De Donder-Weyl equations and get

$$\partial_\mu \mathrm{T}_{ab}^{\mu\nu} = -\mathrm{T}_{c[a}^{\mu\nu}\tau_{b]\nu}^{c} \tag{5.29}$$

$$\partial_\mu \theta^{\mu\nu} = 0 \tag{5.30}$$

$$\partial_\mu \Psi_a^{\mu\nu} = 0 = \frac{1}{4}\mathrm{T}_{ad}^{\mu\sigma}\psi_\sigma^{*d} - \frac{1}{4}\epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}\psi_\nu^{*b}\psi_{[\rho e}^{c}\psi_{\sigma]}^{*ed} \tag{5.31}$$

$$\partial_\mu \Psi_a^{*\mu\nu} = 0 = \frac{1}{4}\mathrm{T}_{ad}^{\mu\sigma}\psi_\sigma^{d} - \frac{1}{4}\epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}\psi_\nu^{b}\psi_{[\rho e}^{c}\psi_{\sigma]}^{*ed}, \tag{5.32}$$

where the last two equations are naturally zero since the associated polymomenta are zero. The EoM are

$$\mathrm{d}_\tau\left(\mathrm{Re}R^{ab} + \psi^a \wedge \psi^{*b}\right)\epsilon_{abcd} = 0 \tag{5.33}$$

$$\mathrm{d}\theta = 0 \tag{5.34}$$

$$\epsilon_{abcd}\left(\mathrm{Re}R^{cd} - 2\psi^c \wedge \psi^{*d}\right) \wedge \psi^{*b} = 0 \tag{5.35}$$

$$\epsilon_{abcd}\left(\mathrm{Re}R^{cd} - 2\psi^c \wedge \psi^{*d}\right) \wedge \psi^{b} = 0, \tag{5.36}$$

where the last two equations are of specific interest for us. We can easily see that, under the substitution $\psi = e + it$, we get

$$\epsilon_{abcd}\Big[\mathrm{Re}R^{cd} - 2\Big(e^c \wedge e^d + t^c \wedge t^d\Big)\Big] \wedge e^b = 0 \quad \text{and}$$
$$\epsilon_{abcd}\Big[\mathrm{Re}R^{cd} - 2\Big(e^c \wedge e^d + t^c \wedge t^d\Big)\Big] \wedge t^b = 0.$$

If we now use the parametrisation $e^a = \overline{e}^a$ around a proportional background $t^a = c\overline{e}^a$ $(c \neq 0)$, the two equations above both reduce to

$$\epsilon_{abcd}\Big[\mathrm{Re}R^{cd} - 2\big(1 + c^2\big)\overline{e}^c\overline{e}^d\Big] \wedge \overline{e}^b = 0, \tag{5.37}$$

which is the same EoM from [10] that we introduced in section 2.3.

We see that the DDW formalism is a valid alternative to the usual approach of the variation of the action, giving us the same solutions and additionally leaving us the freedom to choose at which stage of the calculation we want to break the symmetry. As we mentioned in chapter 4, the DDW equations are equivalent to the Euler-Lagrange equations. On first glance, this fact might seem paradoxical, since the EoM obtained from the Lagrangian differ depending on where during the computation the symmetry is broken down [50], while they are the same if we use the DDW approach. After all, the actions (5.1) and (5.18) are different actions by construction. This contradiction can be cleared up by taking a closer look at the polymomenta. The polymomentum $p_a^{(1)\mu\nu}$ can be interpreted as the variation of $\mathcal{L}$ with regard to the derivative $\partial_\mu e^a_\nu$, "*carrying*" the information about $\frac{\delta\mathcal{L}}{\delta\partial_\mu e^a_\nu}$ with it. We thereby never lose information about this variation during our calculations and during the projection from $SO(1,5)$ down to $SO(1,3) \times SO(2)$. This is not true for the classical variation of the Lagrangian. If the symmetry gets broken down during the construction of the action, information about $\frac{\delta\mathcal{L}}{\delta\partial_\mu e^a_\nu}$ and its influence on the EoM is lost, since it is not part of the Euler-Lagrange equations. On the other hand, if the projection is taken after the variation of the action, the EoM are influenced by the variations of fields which are in the complement of the final symmetry group, therefore yielding a different result.
An interesting question here is whether there is one 'correct' approach and whether there are underlying mathematical or physical justifications for either approach. The obvious first answer here would be that, breaking the symmetry with the construc-

tion of the action led to Einstein's equations in the MM case [5]. Since the projection down to $SO(1,3)\times SO(2)$ or $SO(1,3)$ respectively has been done by hand, there have been attempts to put these processes onto more solid mathematical foundations, for example in the context of Cartan geometry [62].

6 An exploration into constraint analysis of bimetric theory in the De Donder-Weyl formalism

This chapter is an analysis of the constraints of the bimetric theory described in section 1.4 under the lens of the DDW formalism formulated in [12] and introduced in chapter 4. We will start by computing the DDW Hamiltonian for the bimetric theory in four dimensions and the constraints following from this computation. After that we will show a construction of $n-1$-forms from the naturally occurring constraints and explain further steps, like the calculation of all the brackets and the consistency constraints 4.32.

Recall the bimetric action from section 1.4 of the form

$$\mathcal{S}_{HR} = \int \mathrm{d}^4x \left(m_g^2 \sqrt{-g} R(g) + m_f^2 \sqrt{-f} R(f) - 2m^4 \sqrt{-g} \sum_{n=0}^{4} \beta_n e_n(S) \right). \tag{6.1}$$

To compute the DDW Hamiltonian, we will split this action up into the three parts

$$\mathcal{S}_{HR} = \mathcal{S}_{HP}[e,\omega] + \mathcal{S}_{HP}[t,\tilde{\omega}] + \mathcal{S}_{int}[e,t], \tag{6.2}$$

where

$$\mathcal{S}_{HP}[e,\omega] = m_g^2 \int \epsilon_{ABCD} e^A \wedge e^B \wedge \left(R^{CD}(\omega) + \frac{\lambda}{24} e^C \wedge e^D \right) \tag{6.3}$$

is the Hilbert-Palatini action (1.12) for the vielbein e and the spin connection ω. The second action $\mathcal{S}_{HP}[t,\tilde{\omega}]$ takes on the same form as the first and the third action is the interaction term between the fields t and e

$$\mathcal{S}_{int}[e,t] = -4m_g^2 m_f^2 \int \mathrm{d}^4x \epsilon_{ABCD}\epsilon^{\mu\nu\rho\sigma}\Bigg[\frac{\beta_1}{6}e^A_{[\mu}t^B_{\nu]}t^C_{[\rho}t^D_{\sigma]} + \frac{\beta_2}{4}e^A_{[\mu}e^B_{\nu]}t^C_{[\rho}t^D_{\sigma]} + \frac{\beta_3}{6}e^A_{[\mu}e^B_{\nu]}e^C_{[\rho}t^D_{\sigma]}\Bigg]. \tag{6.4}$$

In the above action we used the fact that $\beta_0 = -(3\beta_1 + 3\beta_2 + \beta_3)$ and $\beta_4 = -(\beta_1 + 3\beta_2 + 3\beta_3)$ is a necessary condition for a solution where both metrics constructed from e and t allow flat-space solutions [22]. Following the approach in [22] a general vielbein can be written as a Lorentz boost $\Lambda(p)^A_B$ for any given 3-vector p^a

$$\Lambda^A_B(p) = \begin{pmatrix} \gamma & p^a \\ p_b & \delta^a_b + \frac{1}{\gamma+1}p^a p_b \end{pmatrix}, \tag{6.5}$$

where $\gamma = \sqrt{1 + p_a p^a}$. This gives us a general vielbein, which is an upper-triangular vielbein $\tilde{t}^A_\mu$ of the form (1.9), on which $\Lambda^A_B(p)$ acts. It takes on the form

$$t^A_\mu = \Lambda^A_B(p)\tilde{t}^B_\mu = \begin{pmatrix} M + M^i t^a_i p_a & Mp^b + M^i t^a_i(\delta^a_b \frac{p_a p^a}{\gamma+1}) \\ t^a_i p_a & t^a_i(\delta^b_a + \frac{p_a p^b}{\gamma+1}) \end{pmatrix}.$$

Luckily, due to overall local Lorentz invariance, we are able to set $p^a = 0$ for one of the vielbeins, which we will do for e, s.t. it takes on the form from (1.9), namely

$$e^A_\mu = \begin{pmatrix} N & 0 \\ e^a_j N^j & e^a_i \end{pmatrix}. \tag{6.6}$$

Also, because $\mathcal{S}_{HP}$ is Lorentz invariant we may transform t in such a way that $p^a = 0$ during our treatment of this term.

We can start by computing the polymomenta of the theory and see that the only fields which show up with a derivative are ω and $\tilde{\omega}$ in $R^{AB}(\omega) = \mathrm{d}\omega^{AB} + \omega^A_C \wedge \omega^{CB}$. The polymomenta are therefore straight-forward to calculate giving

$$
\begin{aligned}
\pi^{\mu\nu}_{AB} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \omega^{AB}_\nu)} = \frac{1}{4}\epsilon_{ABCD}\epsilon^{\mu\nu\rho\sigma} e^C_{[\rho} e^D_{\sigma]},\\
\tilde{\pi}^{\mu\nu}_{AB} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \tilde{\omega}\nu^{AB})} = \frac{1}{4}\epsilon_{ABCD}\epsilon^{\mu\nu\rho\sigma} t^C_{[\rho} t^D_{\sigma]},\\
P^{\mu\nu}_A &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu e^A_\nu)} = 0,\\
\tilde{P}^{\mu\nu}_A &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu t^A_\nu)} = 0.
\end{aligned}
\tag{6.7}
$$

From the above equation we can identify two primary constraints, i.e.$C^{\mu\nu}_{(P)A} = P^{\mu\nu}_A \approx 0$ and $C^{\mu\nu}_{(\tilde{P})A} = \tilde{P}^{\mu\nu}_A \approx 0$. These turn, due to the 3+1 reformulation of the vielbeins, into constraints on the polymomenta of N N^i, e^a_i from $C^{\mu\nu}_{(P)A}$ and constraints on the polymomenta of M, M^i, t^a_i and p^a following from $C^{\mu\nu}_{(\tilde{P})A}$.
From the first two equations of (6.7) we obtain the constraints

$$
C^{\mu\nu}_{(\pi)AB} = \pi^{\mu\nu}_{AB} - \frac{1}{4}\epsilon_{ABCD}\epsilon^{\mu\nu\rho\sigma} e^C_{[\rho} e^D_{\sigma]} \approx 0, \tag{6.8}
$$

$$
C^{\mu\nu}_{(\tilde{\pi})AB} = \tilde{\pi}^{\mu\nu}_{AB} - \frac{1}{4}\epsilon_{ABCD}\epsilon^{\mu\nu\rho\sigma} t^C_{[\rho} t^D_{\sigma]} \approx 0. \tag{6.9}
$$

Inspecting the several combinations of space-time and Lorentz indices we also see that the only non-zero polymomenta π and $\tilde{\pi}$ are

$$
\begin{aligned}
\pi^{ij}_{a0} &= -\frac{1}{2}\tilde{\epsilon}_{abc}\tilde{\epsilon}^{ijk} N^l e^b_l e^c_k,\\
\pi^{i0}_{a0} &= \frac{1}{4}\tilde{\epsilon}_{abc}\tilde{\epsilon}^{ijk} e^b_j e^c_k,\\
\pi^{ij}_{ab} &= \frac{1}{2}\tilde{\epsilon}_{abc}\tilde{\epsilon}^{ijk} N e^c_k,
\end{aligned}
$$

where $\tilde{\epsilon}_{abc} \coloneqq \epsilon_{0abc}$.
With all the necessary work done, we are able to write down the DDW Hamiltonian $H_{DW} = \pi^\mu_a \partial_\mu \phi^a - \mathcal{L}$, yielding

$$
\begin{aligned}
H_{DW} = &-\left(\pi^{\mu\nu}_{AB}\omega^A_{[\mu C}\omega^{CB}_{\nu]} + \tilde{\pi}^{\mu\nu}_{AB}\tilde{\omega}^A_{[\mu C}\tilde{\omega}^{CB}_{\nu]}\right)\\
&- MC_{(M)} - M^i C_{(M)i} - NC_{(N)} - N^i C_{(N)i}.
\end{aligned}
\tag{6.10}
$$

The second line in the equation above follows from the $\mathcal{L}_{int}$ part, where the lapse and the shift of both vielbeins turn out to take on the function of Lagrange multipliers, yielding another set of constraints of the form

$$C_{(M)} = \tilde{\epsilon}_{abc}\tilde{\epsilon}^{ijk}\left[\frac{\beta_3}{6}e^a_i e^b_j e^c_k + \frac{\beta_2}{4}e_j e^c_k t^l_i \overline{p}^a_l \frac{\beta_1}{6}\left(3t^l_i \overline{p}^a_l t^e_j \overline{p}^b_e e^c_k + 2t^d_i \overline{p}^b_d t^l_j p_l e^c_k p^a\right)\right] \tag{6.11}$$

$$C_{(M)i} = \tilde{\epsilon}_{abc}\tilde{\epsilon}^{jkl}\left[\frac{\beta_3}{6}t^d_i p_d e^a_j e^b_k e^c_l + \frac{\beta_2}{4}t^d_i p_d t^e_j \overline{p}^a_e e^b_k e^c_l + \frac{\beta_1}{6}\left(3t^d_i p_d t^w_j \overline{p}^a_e t^f_k \overline{p}^b_f e^c_l\right.\right. \tag{6.12}$$

$$\left.\left. +2t^e_i \overline{p}^a_e t^d_j \overline{p}^b_d t^f_k p_f e^c_l\right)\right] \tag{6.13}$$

$$C_{(N)} = \tilde{\epsilon}_{abc}\tilde{\epsilon}^{ijk}\left[\frac{\beta_1}{6}t^e_i \overline{p}^a_e t^l_j \overline{p}^a_l t^f_k \overline{p}^c_f + \frac{\beta_2}{4}t^f_i t^e_j \overline{p}^b_e e^c_k \overline{p}^a_f + \frac{\beta_3}{6}\left(t^a_i e^b_j e^c_k + t^d_i \overline{p}^a_d e^b_j e^c_k\right)\right] \tag{6.14}$$

$$C_{(N)i} = \tilde{\epsilon}_{abc}\tilde{\epsilon}^{jkl}\left[\frac{\beta_1}{6}e^c_i t^d_j p_d t^e_k \overline{p}^a_e t^f_l \overline{p}^b_f + \frac{\beta_2}{4}e^b_i t^d_j p_d e^c_l t^e_k \overline{p}^a_e + \frac{\beta_3}{6}3e^b_i t^d_j p_d e^a_k e^c_l\right], \tag{6.15}$$

where we defined $\overline{p}^b_a \coloneqq (\delta^b_a + \frac{p_a p^b}{1+\gamma})$.

Before we move on and write down the total DDW Hamiltonian, we need to construct $n-1$-forms from the constraints to follow the procedure outlined in section 4.5. We will do this for N and $C_{(N)}$, since the process is the same for every constraint. When constructing our $n-1$ form $\mathcal{C}_{(N)}$, we require that

$$\mathcal{N} \bullet \mathcal{C}_{(N)} = NC_{(N)},$$

causing us to also define a construction for N that yields a 1-form $\mathcal{N}$. We define the simplest $n-1$-form $\mathcal{C}_{(N)\mu} \coloneqq C_{(N)}V_\mu$. To discern the reason why $\mathcal{N}$ is a 1-form, we need to look at the definition of $\bullet$, i.e.

$$\mathcal{N} \bullet \mathcal{C}_{(N)} = \star^{-1}\left(\star\mathcal{N} \wedge \star\mathcal{C}_{(N)}\right).$$

Since $\star$ maps a p into a $n-p$ form (on an n-dimensional space) and we know that NC has degree 0, it is necessary that the term inside the parentheses is an n-form. $\wedge$ maps a p and a q-form into a $p+q$-form, therefore $\mathcal{N}$ needs to be a 1-form. We choose the simplest form for $\mathcal{N}$, namely

$$\mathcal{N} \coloneqq kN\mathrm{d}x^{\mu}, \tag{6.16}$$

where k is just a constant which is inverse to the dimensionality factors that come from resolving $\mathcal{N} \bullet \mathcal{C}_{(N)}$ into NC:

$$\mathcal{N} \bullet \mathcal{C}_{(N)} = \star^{-1}\big(\star\mathcal{N} \wedge \star\mathcal{C}_{(N)}\big) = \star^{-1}\big(\langle\star\mathcal{N}, \mathcal{C}_{(N)}\rangle V\big) \tag{6.17}$$

$$= \frac{1}{(n-1)!} N C_{(N)} \det(\delta^{\nu}_{\mu}) \star^{-1} V = \frac{4}{(n-1)!} N C_{(N)}. \tag{6.18}$$

Where we used the properties of both the wedge product and the Hodge star mentioned in section 1.2 and the fact that $\star^{-1}V = 1$. Since both N and $C_{(N)}$ are not differential forms we were able to take them out of the inner product in the second-to-last equality, leaving only the product $\langle V^{\nu}, V_{\mu}\rangle$. From the above computation we see that $k = \frac{4}{(n-1)!}$.
To help with following calculations, we will calculate some simple brackets from the definition (4.22), yielding

$$\{\pi^{\mu\nu}_{AB}V_{\nu}, \omega^{CD}_{\rho}\} = \delta^{\mu}_{\rho}\delta^{[C}_{A}\delta^{D]}_{B}, \quad \{\pi^{\mu\nu}_{AB}V_{\nu}, \omega^{CD}_{\rho}V_{\sigma}\} = \delta^{\mu}_{\rho}\delta^{[C}_{A}\delta^{D]}_{B}V_{\sigma}, \tag{6.19}$$

$$\{\pi^{\mu\nu}_{AB}, \omega^{CD}_{\rho}\} = \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma}\delta^{[C}_{A}\delta^{D]}_{B}, \quad \{\psi^{\mu}V_{\mu}, N\} = 1, \quad \{\psi^{\mu}V_{\mu}, NV_{\nu}\} = V_{\nu} \tag{6.20}$$

$$\{\psi^{\mu}, NV_{\nu}\} = \delta^{\mu}_{\nu}, \quad \{\psi^{\mu}_{a}V_{\mu}, N^{b}\} = \delta^{b}_{a}, \quad \{\psi^{\mu}_{a}V_{\mu}, N^{b}V_{\nu}\} = \delta^{b}_{a}V_{\nu} \tag{6.21}$$

$$\{\psi^{\mu}_{a}, N^{b}V_{\nu}\} = \delta^{b}_{a}\delta^{\nu}_{\mu}, \quad \{P^{\mu i}_{a}V_{\mu}, e^{b}_{j}\} = \delta^{b}_{a}\delta^{i}_{j}, \quad \{P^{\mu i}_{a}V_{\mu}, e^{b}_{j}V_{\sigma}\} = \delta^{b}_{a}\delta^{i}_{j}V_{\sigma}, \tag{6.22}$$

$$\{P^{\mu i}_{a}, e^{b}_{j}V_{\sigma}\} = \delta^{b}_{a}\delta^{i}_{j}\delta^{\mu}_{\sigma}. \tag{6.23}$$

Where we defined the polymomenta $\psi^{\mu} \coloneqq \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}N)}$, $\psi^{\mu}_{a} \coloneqq \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}N^{a})}$ and $P^{\mu i}_{a} \coloneqq \frac{\partial\mathcal{L}}{\partial\partial_{\mu}e^{a}_{i}}$.
It is good to know here, that the multivector field corresponding to the $n-1$ forms constructed from the polymomenta are just the negative partial derivatives by their corresponding fields, which can be shown by a straight-forward calculation starting out from (**??**). The above calculation is mirrored for the other polymomenta pairs $M \leftrightarrow \chi^{\mu}$, $M^{a} \leftrightarrow \chi^{\mu}_{a}$ and $t^{a}_{i} \leftrightarrow \tilde{P}^{\mu i}_{a}$ and $p^{a} \leftrightarrow \theta_{a}$. Also note that, by definition, the only non-vanishing bracket of a field is with its polymomenta.
The total DDW Hamiltonian can be expressed as

$$\begin{aligned}\tilde{H}_{DW} = H_{DW} &+ \lambda^{AB}_{\mu} \bullet \mathcal{C}^{\mu}_{(\pi)AB} + \tilde{\lambda}^{AB}_{\mu} \bullet \mathcal{C}^{\mu}_{(\tilde{\pi})AB} \\ &+ \tau^a_i \bullet P^i_a + \tilde{\tau}^a_i \bullet \tilde{P}^i_a + \xi \bullet \psi + \tilde{\xi} \bullet \chi \\ &+ \xi^i \bullet \psi_i + \chi^i \bullet \chi_i + \kappa^a \bullet \theta_a,\end{aligned} \tag{6.24}$$

where the construction (6.16) is implied for all constraints and multipliers.
The next step is to look for secondary constraints following from the consistency condition (4.32), i.e.

$$\mathbf{d} \bullet \mathcal{C}_m = \{\tilde{H}_{DW}, \mathcal{C}_m\} = \{H_{DW}, \mathcal{C}_m\} + u^n \bullet \{\mathcal{C}_n, \mathcal{C}_m\} \approx 0. \tag{6.25}$$

Since we know that a lot of the constraints can be used interchangeably by just exchanging the corresponding components of e and t and/or ω and $\tilde{\omega}$, it is not necessary to write down all results from this computation. The calculation of the condition $\mathrm{d} \bullet \psi \approx 0$ will be done in some detail, to give the reader an idea of the steps necessary. The condition for ψ^μ, the polymomentum of N, takes on the form

$$\begin{aligned}\mathrm{d} \bullet \psi = \{\tilde{H}_{DW}, \psi^\nu V_\nu\} &= \lambda^{AB}_{\mu} \bullet \{\mathcal{C}^{\mu}_{(\pi)AB}, \psi^\nu V_\nu\} \\ &= \lambda^{AB}_{\mu} \bullet \{-\epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD} e^C_{[\rho} e^D_{\sigma]} V_\nu, \psi^\delta V_\delta\},\end{aligned}$$

where the second-to-last equality is a result of the fact, that the only constraint containing N, which gives a non-vanishing bracket is $\mathcal{C}^{\mu}_{(\pi)AB}$. There is obviously the N in $\mathcal{N} \bullet \mathcal{C}_{(N)}$, but that term does not impact the result, since it is multiplied with a constraint which is always zero on the constraint surface. Since the terms in the bracket commute and we know that $X_\psi = -\frac{\partial}{\partial N}$, we can use the definition of the bracket and get

$$
\begin{aligned}
&\lambda^{AB}_{\mu}\left((-1)^{n-(n-1)}(-\frac{\partial}{\partial N})\lrcorner \mathrm{d}^V\left(-\epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD}e^C_{[\rho}e^D_{\sigma]}V_\nu\right)\right)\\
&=\lambda^{AB}_{\mu}\left(-(\frac{\partial}{\partial N})\lrcorner\frac{\partial}{\partial e^E_\delta}\left(-\epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD}e^C_{[\rho}e^D_{\sigma]}V_\nu\right)\mathrm{d}e^E_\delta\right)\\
&=-\lambda^{AB}_{\mu}\bullet\left(\frac{\partial}{\partial N}(\epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD}e^C_{[\rho}e^D_{\sigma]}V_\nu)\right)\\
&=-\lambda^{AB}_{\mu}\bullet(\epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD}e^C_{\rho}V_\nu)\approx 0
\end{aligned}
$$

The last line does neither evaluate to identically zero nor put a constraint on λ^{AB}_{μ}, therefore it is safe to assume that we have found a secondary constraint $\mathcal{C}^{(2)}\approx 0$. With this calculation we have actually found another constraint. We know that $\{\tilde{H}_{DW},\chi^\nu V_\nu\}$ takes on the same form, if we only replace the components of e with the corresponding components of t.

The above analysis has been done for more constraints and can be found in section 3. Next steps would be the completion of the above analysis for all constraints, the identification of further secondary constraints from the consistency conditions and classification of said constraints into first-class and second-class. From there on, one would hypothetically be able to count the constraints on the theory and judge whether the above analysis yields the correct number of constraints to propagate neither too many nor too few DoF. Even though there wasn't any literature available, proving the hypothesis that first-class and second-class constraints remove the same amount of DoF in the DDW as they do in the traditional Hamiltonian formalism, one could make a good argument for such an idea due to the stark similarities between the two formulations.

7 Conclusion and Outlook

This thesis was devoted to introducing the DDW formalism in the context of spin-2 field theories. We started out by outlining the important features of the most prominent spin-2 theories in chapter 1 and introduced several gauge theoretic approaches to a spin-2 theory in chapter 2. In chapter 4 we gave a high-level introduction to the formalism and applied it to the simple example of scalar fields in vacuum.
In chapter 5 we applied the DDW formalism to the $SO(1,5)$ gauge theory of bimetric theory from [10], which we introduced in section 2.3. The theory is based on the approach by MacDowell and Mansouri [5] and follows the same approach, where the symmetry of the action is broken down to the subgroup $SO(1,3) \times SO(2)$. In the aforementioned approach, the symmetry is broken by construction of the action. In section 5.1 we showed that, we would get the same EoM, independent of whether we followed the construction from [10] or used a general action with the complete $SO(1,5)$ action and took the projection into $SO(1,3) \times SO(2)$ after the calculation of the EoM.
We also computed the EoM of the $SO(1,5)$ theory with the complex vielbein
$\psi = e + it$ and came to the same results as the original works in [10].
The stand-out feature of the DDW formalism is its covariance. It allowed us to keep the calculations more concise by not having to treat time derivatives separate. Another interesting advantage of this formalism is the fact that the polymomenta 'encode' more information than the canonical momenta of the Hamiltonian formalism. This allowed us to obtain the same results, even if we took the projection into $SO(1,3) \times SO(2)$ in differing ways. Unlike in the MM approach, where we lose information by requiring the curvatures of e and t to vanish [50], the information about the curvatures "travels through the calculation" with the corresponding polymomentum. The next big question here is, whether the covariant preservation of this information is just an interesting feature of the formalism or a powerful new tool that a physicist can use to explore new theories.

In chapter 6 we performed an constraint analysis of the $SO(1,5)$ theory. Unlike

the computation done in chapter chapter 5, the constraint analysis has been quite cumbersome compared to the traditional analysis of a Hamiltonian and its constraints. The construction of a constraint and its multiplier into a $n-1$ and a 1 form as well as the entire bracket formalism require deep knowledge of differential geometry and do not deliver any obvious advantages to the traditional formalism.
On the other hand, the work done in chapters chapter 4, chapter 5 and chapter 6 are based on the works [12, 46, 53, 54, 55, 56, 57] and [58], to name a few, which are all quite technical. This is not a bad thing, but one of the most interesting questions regarding the DDW formalism is, whether it could be a valid alternative to the traditional Hamiltonian formalism in the analysis of covariant theories like GR. Obviously, the formalism stands and falls by the results obtained from calculations using the DDW formalism. However, assuming the formalism was giving valid results, one can wonder, whether this is something one could teach a graduate student (this thesis is, after all, meant to be understandable on a graduate student level). The works above are technical for a reason and that is, giving a general framework to treat a wide variety of physical and mathematical problems. Were one to reformulate this theory in the context of the special case of spin-2 fields in a 4 dimensional manifold describing spacetime it could be possible to teach a 'light' version of the DDW formalism which could give valuable insights to graduate students, after all, when we were undergraduate students in our first semester and learned about the Hamiltonian, we were never introduced to symplectic manifolds and such.

Although the DDW formalism has been around for almost a century it has only recently been gaining in popularity and the pool of understanding around it is still growing. Nonetheless, it seems, due to it's strong geometrical foundation in differential geometry, a quite natural tool for the exploration of inherently geometrical theories like the spin-2 field theories covered in this thesis. It will be interesting to observe future developments in this field and see where the scientific community will take it.

Appendix

1 Mathematical Definitions

1.1 Fiber Bundles, Tangent Bundles and more

Before introducing *tangent bundles*, it is necessary to become familiar with *fiber bundles*. A fiber bundle is a space X with a projection map $\pi : X \to M$ which can locally be represented as a product space $M \times F$. We call X the *total space*, M *base space* and F is called the *fiber* of X. It is not a necessity that X can be globally represented as a product bundle. For example, both a cylinder and a Mobius strip may be interpreted as fiber bundles X, with a closed circle around the surface being the base space and straight lines projecting in right angles from this ring being the fibers F.
The tangent bundle

$$TX = \bigcup_{x \in X} T_x X = \{(x, y) | x \in X, y \in T_x X\}$$

is a *vector bundle*, namely a fiber bundle whose fibers are vector spaces. The tangent bundle is equipped with the natural projection $\pi(x, y) = x$. We call the projection $\phi : M \to X$ with $\phi(\pi(x)) = x$ for all $x \in M$ a *section* of the fiber bundle M. The *vertical bundle* and the *horizontal bundle* are two subbundles of a tangent bundle of a smooth fiber bundle. The vertical bundle $T^V X$ is the *kernel* of the projection map $T\pi : TX \to TM$, i.e. the map to the zero vector on TM. Constricting $T\pi$ to just the subspace $T^V X \subset TX$ yields a new bundle, called a *subbundle* of TX. The horizontal bundle is then a choice of subbundle of TX, such that at each point $x \in X$ we have $T_x X = T_x^V X \times T_x^H X$. A vector in $T_X^{V/H}$ is called *vertical/horizontal.*

1.2 The exterior algebra

To understand this section, like the entirety of this thesis we expect a basic understanding of manifolds, vectors, and similar mathematical constructs.
Given a vector space V and its dual V^*, we can define their exterior algebras $\Lambda(V)$ and $\Lambda(V^*)$. The product on $\Lambda(V)$ is the *wedge product*, or exterior product, which we will define below. A *differential p-form* is an element of $\Lambda^p(V^*) \subset \Lambda(V^*)$, the space of all p-forms in $\Lambda(V^*)$, and has the form

$$\omega^{(p)} := \omega_{\mu_1 \cdots \mu_p}(x) \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p},$$

where $\mathrm{d}x^\mu$ is a coordinate differential and $\omega_{\mu_1 \cdots \mu_p}$ an antisymmetric tensor of rank p ($p \in 1, \cdots, n$, where n is the dimension of the base manifold M). The wedge product $\wedge$ is the totally antisymmetric product of differential forms, e.g. for two 1-forms it becomes

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}(\mathrm{d}x^\mu \mathrm{d}x^\nu - \mathrm{d}x^\nu \mathrm{d}x^\mu).$$

The wedge product is essential to working with forms and maps p and a q form to a $p+q$ form as long as $p+q \leq n$, otherwise it vanishes. Th *exterior derivative* d of a p-form is defined as

$$\mathrm{d}\omega^{(p)} := \frac{1}{p!} \partial_\mu \omega_{\mu_1 \cdots \mu_p}(x) \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$$

and maps a p-form into a $p+1$-form.
In the same way as for covectors, the wedge product can be defined for vectors. The product of several vectors is called a *multivector*. The wedge product of two vectors (a bivector) is of the form

$$\partial_\mu \wedge \partial_\nu = \frac{1}{2}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu).$$

Under closer inspection we can find that p and q forms have the same amount of components, iff $p+q = n$. This makes it possible to define a one-to-one map between them, which is called the *Hodge dual* or *Hodge star* and is denoted by $\star$. It has the property $\omega^{(p)} \wedge \star\omega^{(q)} = \langle\omega^{(p)}, \omega^{(q)}\rangle V$, with $V := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, which defines it completely. For some examples look at

$$\star 1 = V, \tag{1}$$

$$\star\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z \qquad \text{for } n = 3, \tag{2}$$

$$\star(\star\omega^{(p)}) = (-1)^{\sigma}(-1)^{pq}\omega^{(p)}, \tag{3}$$

where in the last example $\sigma = 0$ for a metric with Euclidean signature and $\sigma = 1$ if has a Lorentzian signature.

1.3 The interior product

The interior product $\lrcorner$ is defined as a contraction between a differential for ω and a vector field X. It is sometimes called *interior derivative* or *antiderivative of degree -1* and can also be found denoted by ι_X. It is the map

$$X\lrcorner : \Omega^p(M) \to \Omega^{p-1}(M),$$

that maps a p-form ω to the $p-1$- form $X\lrcorner\omega$. It is defined by requiring the action of $X\lrcorner\omega$ on a vector Y being

$$(X\lrcorner\omega)(Y) = \omega(X, Y),$$

in other words the interior product can be seen as fixing the first argument of a p-form to X, s.t. it becomes a $p-1$-form through contraction.

1.4 The Poincaré - Cartan form

In [45] Cartan expresses the variation of the action $\delta\mathcal{S} = \mathrm{d}\mathcal{S}(Y)$ using a variation vector field $Y(\epsilon, t)$ with ϵ parametrising the variation. Choosing the vector field Y in a way that it vanishes on the boundaries $t_i(\epsilon)$ of the variation, Cartan got the expression

$$\delta\mathcal{S} = \left[\frac{\partial\mathcal{L}}{\partial\dot{q}}\frac{\partial q_\epsilon}{\partial\epsilon}\bigg|_{\epsilon=0} + \mathcal{L}\mathrm{d}t(Y)\right]_{t_0(\epsilon)}^{t_1(\epsilon)} - \int_{t_0(\epsilon)}^{t_1(\epsilon)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q}\right]\frac{\partial q_\epsilon}{\partial\epsilon}\bigg|_{\epsilon=0}\mathrm{d}t,$$

where $q_\epsilon = q(\epsilon, t)$ is the varied curve. The second term vanishes because of the Euler-Lagrange equations and Cartan continues his discussion inspecting the variable boundary conditions depending on ϵ. The first term can be expressed as just

$$\delta\mathcal{S} = \left[\frac{\partial\mathcal{L}}{\partial\dot{q}}\mathrm{d}q - \left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\dot{q} - \mathcal{L}\right)\mathrm{d}t\right]_{t_0(\epsilon)}^{t_1(\epsilon)}(Y) = [\Theta]_{t_0(\epsilon)}^{t_1(\epsilon)}(Y) = \Theta_1 - \Theta_0,$$

where we introduced the *Poincaré - Cartan form*

$$\Theta = p\mathrm{d}q - H\mathrm{d}t$$

In the next step Cartan considers a tube formed by real trajectories between the interval $[t_0, t_1]$ in phase space in the form of figure 1. An integration around such a closed loop, parametrized by ϵ is then just

$$\oint \mathcal{S} = \oint \Theta_1 = \oint \Theta_2 = 0.$$

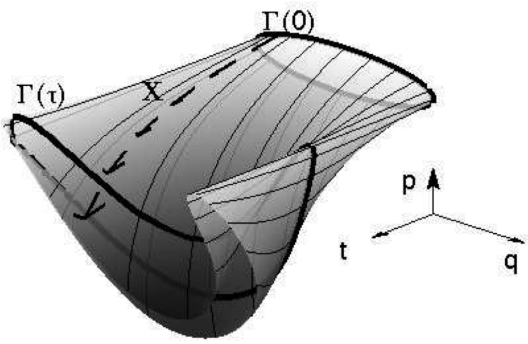

Figure 1: Two closed loops T(τ) and T(0) around a collection of real trajectories with a tangent vector field X [63].

This means, that the quantity $I = \oint \Theta$ is invariant of the choice of loop or in other words, it is invariant under transportation generated by a vector field X tangent to the real trajectories. In mathematical terms this can be expressed as $L_X I = \oint L_X \Theta = 0$, where L_X is the *Lie derivative* along X. Note that Θ is a differential form and the Lie derivative of a differential form is given by $L_X\Theta = X \lrcorner \mathrm{d}\Theta + \mathrm{d}(X \lrcorner \Theta)$, yielding the relation

$$0 = \oint X \lrcorner \mathrm{d}\Theta \qquad \Rightarrow \qquad X \lrcorner \mathrm{d}\Theta = 0.$$

The 2-form

$$\Omega = \mathrm{d}\Theta = \mathrm{d}q \wedge \mathrm{d}p + \mathrm{d}H \wedge \mathrm{d}t,$$

is called the Poincaré - Cartan 2-form or (pre)-symplectic form. Substituting the above definition into condition $X \lrcorner \Omega = 0$, we get

$$(\mathrm{d}H - \frac{\partial H}{\partial t}\mathrm{d}t) X \lrcorner \mathrm{d}t - (\mathrm{d}p + \frac{\partial H}{\partial q}\mathrm{d}t) X \lrcorner \mathrm{d}q + (\mathrm{d}q - \frac{\partial H}{\partial p}\mathrm{d}t) X \lrcorner p = 0.$$

Going term by term we obtain the Hamiltonian EoM from the second and the third term, giving us

$$\dot{p} = -\frac{\partial H}{\partial q} \qquad \text{and} \qquad \dot{q} = \frac{\partial H}{\partial p}.$$

The first term then just yields the variation of the Hamiltonian in time, which is 0 if it does not explicitly depend on t. We see, that all the information about the dynamics of the system are contained within the PC form.

2 Calculations from Chapter 5

2.1 Equations of motion for real vielbeins

The first line in 5.7 for the fields e^a_μ, t^a_μ and A_μ yields

$$\partial_\mu e^a_\nu = -\frac{1}{8}\epsilon_{\mu\nu\rho\sigma}\eta^{ab}p_b^{(1)\rho\sigma} - \frac{1}{2}\Big(\omega^a_{[\mu c}e^c_{\nu]} + t^a_{[\mu}A_{\nu]}\Big) \tag{4}$$

$$\partial_\mu t^a_\nu = -\frac{1}{8}\epsilon_{\mu\nu\rho\sigma}\eta^{ab}p_b^{(2)\rho\sigma} - \frac{1}{2}\Big(\omega^a_{[\mu c}t^c_{\nu]} + e^a_{[\mu}A_{\nu]}\Big) \tag{5}$$

$$\partial_\mu A_\nu = -\frac{1}{8}\epsilon_{\mu\nu\rho\sigma}\theta^{\rho\sigma} + \frac{1}{2}e_{[\mu c}t^c_{\nu]}. \tag{6}$$

The second line gives the relations for the polymomenta, i.e.

$$\partial_\mu p_a^{(1)\mu\nu} = \frac{1}{2}\Big(\omega^c_{a\mu}p_c^{(1)\mu\nu} + t_{\mu a}\theta^{\mu\nu} + e^c_\mu\pi^{\mu\nu}_{ac} - A_\mu p_a^{(2)\mu\nu}\Big) \tag{7}$$

$$\partial_\mu p_a^{(2)\mu\nu} = \frac{1}{2}\Big(\omega^c_{a\mu}p_c^{(2)\mu\nu} - e_{\mu a}\theta^{\mu\nu} + e^c_\mu\pi^{\mu\nu}_{ac} + A_\mu p_a^{(1)\mu\nu}\Big) \tag{8}$$

$$\partial_\mu\theta^{\mu\nu} = -\frac{1}{2}\Big(e^a_\mu p_a^{(2)\mu\nu} - t^a_\mu p_a^{(1)\mu\nu}\Big). \tag{9}$$

Resolving the above equations for the polymomenta yields

$$p_a^{(1)\mu\nu} = -\epsilon^{\mu\nu\rho\sigma}\eta_{ab}\Big(2\partial_{[\rho}e^b_{\sigma]} + \omega^b_{[\rho c}e^c_{\sigma]} - t^b_{[\rho}A_{\sigma]}\Big)$$

$$p_a^{(2)\mu\nu} = -\epsilon^{\mu\nu\rho\sigma}\eta_{ab}\Big(2\partial_{[\rho}t^b_{\sigma]} + \omega^b_{[\rho c}t^c_{\sigma]} + t^b_{[\rho}A_{\sigma]}\Big)$$

$$\theta^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\Big(e_{[\rho c}t^c_{\sigma]} - 2\partial_\rho A_\sigma\Big),$$

from which we then can compute the curvatures specified in 5.15.

3 Calculations from Chapter 6

Following the approach in the main chapter, we make use of the symmetry between the components of e and t. We start by computing the bracket for the constraint $\mathcal{C}_{(\psi)i} \approx 0$

$$
\begin{aligned}
\{\tilde{H}_{DW}, \psi_i^\mu V_\mu\} &= \lambda_\mu^{AB} \bullet \{\psi_i^\delta V_\delta, \epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD} V_\nu e^C_{[\rho} e^D_{\sigma]}\} \\
&= -\lambda_\mu^{AB} \bullet \left(\frac{\partial}{\partial N^i} \lrcorner \mathrm{d}^V \left(\epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD} V_\nu e^C_{[\rho} e^D_{\sigma]} \right) \right) \\
&= -\lambda_\mu^{AB} \bullet \left(2\epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD} V_\nu e^C_{[\rho} \frac{\partial e^D_{\sigma]}}{\partial N^i} \right) \\
&= -\lambda_\mu^{AB} \bullet \left(2\epsilon^{\mu\nu\rho 0}\epsilon_{ABCd} V_\nu e^C_\rho e^d_i \right) \\
&= -\lambda_j^{AB} \bullet \left(2\tilde{\epsilon}^{jkl}\epsilon_{ABCd} V_k e^C_l e^d_i \right) \approx 0,
\end{aligned}
$$

which gives a secondary constraint $C^{(2)}_{(N)i}$ to be analysed further.
For $\mathcal{C}_{(p)a} \approx 0$ we get

$$
\begin{aligned}
\{\tilde{H}_{DW}, \theta_a^\mu V_\mu\} &= \{\theta_a^\mu V_\mu, \tilde{H}_{DW}\} \\
&= \frac{\partial}{\partial p^a} \lrcorner \mathrm{d}^V \tilde{H}_{DW}
\end{aligned}
$$

The only non-vanishing part of $(-\frac{\partial}{\partial p^a}) \lrcorner \mathrm{d}^V \tilde{H}_{DW}$ is

$$
\begin{aligned}
\frac{\partial}{\partial p^a} \tilde{H}_{DW} = \mathcal{N}^\mu \bullet \frac{\partial}{\partial p_a} \mathcal{C}_{(N)\mu} + \mathcal{N}^{\mu i} \bullet \frac{\partial}{\partial p_a} \mathcal{C}_{(N)\mu i} \\
+ \mathcal{M}^\mu \bullet \frac{\partial}{\partial p_a} \mathcal{C}_{(M)\mu} + \mathcal{M}^{\mu i} \bullet \frac{\partial}{\partial p_a} \mathcal{C}_{(M)\mu i},
\end{aligned}
$$

which we can then compute term by term. The results are

$$
\begin{aligned}
\frac{\partial}{\partial p_a} \mathcal{C}_{(N)\mu} &= \frac{V_\mu}{2} \tilde{\epsilon}_{bcd} \tilde{\epsilon}^{ijk} t_i^g \overline{P}^b_{ag} t^e_j \overline{p}^d_e t^f_k \overline{p}^d_f, \\
\frac{\partial}{\partial p_a} \mathcal{C}_{(N)\mu i} &= \frac{V_\mu}{6} \tilde{\epsilon}_{bcd} \tilde{\epsilon}^{jkl} e^d_i \left(t^g_j \eta_{ga} t^e_k \overline{p}^b_e t^f_l \overline{p}^c_f + 2 t^e_j p_d t^e_k \overline{P}^b_{ie} t^f_k \overline{p}^c_f \right).
\end{aligned}
$$

For the constraints $\mathcal{C}_{(\pi)AB}^{\mu\nu} = \pi^{\mu\nu}_{AB} - \frac{1}{4}\epsilon_{ABCD}\epsilon^{\mu\nu\rho\sigma} e^C_{[\rho} e^D_{\sigma]} \approx 0$ (as well as the constraint resulting from $\tilde{\pi}$), we get

$$
\begin{aligned}
&\{\tilde{H}_{DW}, (\pi^{\mu\nu}_{AB} - \frac{1}{4}\epsilon_{ABCD}\epsilon^{\mu\nu\rho\sigma} e^C_{[\rho} e^D_{\sigma]}) V_\nu\} \\
&= \{\pi^{\mu\nu}_{AB} V_\nu, \pi^{\rho\sigma}_{CD}\omega^E_{[\rho C}\omega^{ED}_{\sigma]}\} + \frac{1}{8}\tau^a_i \bullet \{P^i_a, \epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD} e^C_{[\rho} e^D_{\sigma]}\} \\
&= \frac{\partial}{\partial \omega^{AB}_\mu}(\pi^{\rho\sigma}_{CD}\omega^C_{[\rho E}\omega^{ED}_{\sigma]}) + \frac{1}{4}\tau^a_i \epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD} V_\nu \frac{\partial}{\partial e^a_i}(e^C_{[\rho} e^D_{\sigma]}) \approx 0,
\end{aligned}
$$

which is an equation for τ^a_i and therefore not a secondary constraint.
The constraints $\mathcal{C}_{(N)\mu}$, $\mathcal{C}_{(N)i\mu}$, $\mathcal{C}_{(M)\mu}$ and $\mathcal{C}_{(M)i\mu}$ give the relations

$$
\begin{aligned}
\{\mathcal{C}_{(N)\mu}, \tilde{H}_{DW}\} &= -\tilde{\tau}^a_i \bullet \{\tilde{P}^i_a, \mathcal{C}_{(N)\mu}\} - \kappa^a \bullet \{\theta_a, \mathcal{C}_{(N)\mu}\} \\
\{\mathcal{C}_{(N)i\mu}, \tilde{H}_{DW}\} &= -\tilde{\tau}^a_j \bullet \{\tilde{P}^j_a, \mathcal{C}_{(N)i\mu}\} - \kappa^a \bullet \{\theta_a, \mathcal{C}_{(N)i\mu}\} \\
\{\mathcal{C}_{(M)\mu}, \tilde{H}_{DW}\} &= -\tau^a_i \bullet \{P^i_a, \mathcal{C}_{(M)\mu}\} - \tilde{\tau}^a_i \bullet \{\tilde{P}^i_a, \mathcal{C}_{(M)\mu}\} - \kappa^a \bullet \{\theta_a, \mathcal{C}_{(M)\mu}\} \\
\{\mathcal{C}_{(M)i\mu}, \tilde{H}_{DW}\} &= -\tau^a_i \bullet \{P^i_a, \mathcal{C}_{(M)i\mu}\} - \tilde{\tau}^a_i \bullet \{\tilde{P}^i_a, \mathcal{C}_{(M)i\mu}\} - \kappa^a \bullet \{\theta_a, \mathcal{C}_{(M)i\mu}\},
\end{aligned}
$$

which haven't been computed completely. The first two brackets take on the shapes

$$
\begin{aligned}
\{\mathcal{C}_{(N)\mu}, \tilde{H}_{DW}\} &= -\tilde{\tau}^a_i \bullet \left(\frac{V_\mu}{2}\tilde{\epsilon}_{bcd}\tilde{\epsilon}^{ijk}\overline{p}^b_a t^e_j \overline{p}^c_e t^g_k \overline{p}^d_g\right) - \kappa^a \bullet \left(\frac{V_\mu}{2}\tilde{\epsilon}_{bcd}\tilde{\epsilon}^{ijk} t^e_i P^b_{ad} t^f_j \overline{p}^b_f t^g_k \overline{p}^c_g\right) \\
\{\mathcal{C}_{(N)i\mu}, \tilde{H}_{DW}\} &= -\tilde{\tau}^a_j \bullet \left(\frac{V_\mu}{6}\tilde{\epsilon}_{bcd}\tilde{\epsilon}^{jkl} e^d_i \left(p_a t^e_k \overline{p}^b_e t^g_l \overline{p}^c_g - 2t^e_k p_e \overline{p}^b_a t^g_l \overline{p}^c_g\right)\right) \\
&\quad - \kappa^a \bullet \left(\frac{V_\mu}{6}\tilde{\epsilon}_{bcd}\tilde{\epsilon}^{jkl} e^d_i \left(t^e_j \eta_{ea} t^f_k \overline{p}^b_f t^g_l \overline{p}^c_g + 2t^e_j p_e t^f_k P^b_{af} t^g_l \overline{p}^c_g\right)\right).
\end{aligned}
$$

The bracket $\{P^i_a, \tilde{H}_{DW}\}$ evaluates to

$$
- \mathcal{M}^\mu \bullet \{P^i_a, \mathcal{C}_{(M)\mu}\} - \mathcal{M}^{\mu l}\{P^i_a, \mathcal{C}_{(M)l\mu}\} + \lambda^{AB}_\mu \bullet \{P^i_a, \mathcal{C}^\mu_{(\pi)AB}\} \tag{10}
$$

$$
= -\mathcal{M}^\mu \bullet \left(V_\mu \tilde{\epsilon}_{abc}\tilde{\epsilon}^{ijk}\left(\frac{1}{2}t^d_j \overline{p}^b_d t^e_k \overline{p}^c_e + \frac{1}{3}t^d_j \overline{p}^c_d t^e_k p_e p^d\right)\right) \tag{11}
$$

$$
- \tilde{M}^{\mu l} \bullet \left(V_\mu \tilde{\epsilon}_{abc}\tilde{\epsilon}^{ijk}\left(\frac{1}{2}t^d_l p_d t^e_j \overline{p}^b_e t^f_k \overline{p}^c_f + \frac{1}{3}t^d_l \overline{p}^b_d t^a_j \overline{p}^c_e t^f_k p_f\right)\right) \tag{12}
$$

$$
+ \lambda^{AB}_\mu \bullet \left(2V_\nu \epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD}\left(\frac{\partial}{\partial e^a_i} e^C_{[\rho}\right) e^D_{\sigma]}\right). \tag{13}
$$

Then we also have the brackets with $\tilde{P}$ and θ, which are

$$
\begin{aligned}
\{\tilde{P}^i_a, \tilde{H}_{DW}\} = -\Big(&\mathcal{M}^\mu \bullet \{\tilde{P}^i_a, \mathcal{C}_{(M)\mu}\} + \mathcal{M}^{\mu l} \bullet \{\tilde{P}^i_a, \mathcal{C}_{(M)l\mu}\} \\
&+ \mathcal{N}^\mu \bullet \{\tilde{P}^i_a, \mathcal{C}_{(N)\mu}\} + \mathcal{N}^{\mu l} \bullet \{\tilde{P}^i_a, \mathcal{C}_{(N)l\mu}\} - \tilde{\lambda}^{AB}_\mu \bullet \{\tilde{P}^i_a, \mathcal{C}^\mu_{(\tilde{\pi})AB}\}\Big) \\
\{\theta_a, \tilde{H}_{DW}\} = --\Big(&\mathcal{M}^\mu \bullet \{\theta_a, \mathcal{C}_{(M)\mu}\} + \mathcal{M}^{\mu l} \bullet \{\theta_a, \mathcal{C}_{(M)l\mu}\} \\
&+ \mathcal{N}^\mu \bullet \{\theta_a, \mathcal{C}_{(N)\mu}\} + \mathcal{N}^{\mu l} \bullet \{\theta_a, \mathcal{C}_{(N)l\mu}\}\Big)
\end{aligned}
$$

and are still left to be evaluated completely.

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